Proceedings of the ASME 2011 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2011
August 29-31, 2011, Washington, DC, USA

DETC2011-47652

APPLICATION OF A BAYESIAN FILTER TO ESTIMATE UNKNOWN HEAT FLUXES IN A NATURAL CONVECTION PROBLEM

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ABSTRACT

Sequential Monte Carlo (SMC) or Particle Filter Methods, which have been originally introduced in the beginning of the 50's, became very popular in the last few years in the statistical and engineering communities. Such methods have been widely used to deal with sequential Bayesian inference problems in fields like economics, signal processing, and robotics, among others. SMC Methods are an approximation of sequences of probability distributions of interest, using a large set of random samples, named particles. These particles are propagated along time with a simple Sampling Importance distribution. Two advantages of this method are: they do not require the restrictive hypotheses of the Kalman filter, and can be applied to nonlinear models with non-Gaussian errors. This papers uses a SMC filter, namely the ASIR (Auxiliary Sampling Importance Resampling Filter) to estimate a heat flux in the wall of a square cavity undergoing a natural convection. Measurements, which contain errors, taken at the boundaries of the cavity are used in the estimation process. The mathematical model, as well as the initial condition, are supposed to have some error, which are taken into account in the probabilistic evolution model used for the filter.

NOMENCLATURE

C_p specific heat at constant pressure
H height of the cavity
K thermal conductivity
$\mathbf{n}$ measurement noise vector
P pressure
q heat flux
T temperature
T_c "cold" temperature
T_h "hot" temperature
T_{ref} reference temperature
u, v velocity field components in the x and y directions
$\mathbf{v}$ state noise vector
w weights used in the posterior density function
W width of the cavity
x, y Cartesian coordinates
$\mathbf{x}$ state variable vector
$\mathbf{z}$ observation vector
β thermal expansion coefficient
μ viscosity
ρ density

INTRODUCTION

State estimation problems, also designated *as nonstationary inverse problems* [1], are of great interest in innumerable practical applications. In such kinds of problems, the available measured data is used together with prior knowledge about the physical phenomena and the measuring devices, in order to sequentially produce estimates of the desired dynamic variables. This is accomplished in such a manner that the error is minimized statistically [2]. For example, the position of an aircraft can be estimated through the time-integration of its velocity components since departure. However, it may also be measured with a GPS system and an altimeter. State estimation problems deal with the combination

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of the model prediction (integration of the velocity components that contain errors due to the velocity measurements) and the GPS and altimeter measurements that are also uncertain, in order to obtain more accurate estimations of the system variables (aircraft position).

State estimation problems are solved with the so-called Bayesian filters [1,2]. In the Bayesian approach to statistics, an attempt is made to utilize all available information in order to reduce the amount of uncertainty present in an inferential or decision-making problem. As new information is obtained, it is combined with previous information to form the basis for statistical procedures. The formal mechanism used to combine the new information with the previously available information is known as Bayes' theorem [1,3].

The most widely known Bayesian filter method is the Kalman filter [1,2,4-9]. However, the application of the Kalman filter is limited to linear models with additive Gaussian noises. Extensions of the Kalman filter were developed in the past for less restrictive cases by using linearization techniques [1,3,6,7,8]. Similarly, Monte Carlo methods have been developed in order to represent the posterior density in terms of random samples and associated weights. Such Monte Carlo methods, usually denoted as particle filters among other designations found in the literature, do not require the restrictive hypotheses of the Kalman filter. Hence, particle filters can be applied to non-linear models with non-Gaussian errors [1,4,8-17].

Hammersley and Hanscomb [18] presented a technique that used recursive Bayesian filters, together with Monte Carlo simulations, known as Sequential Importance Sampling (SIS). In such approach, the key idea was to represent the posteriori probability function as a set of random samples associated with some weights, in order to calculate the estimates based on such samples and weights. Gordon et al [19] added an extra step, named re-sampling, into the Sequential Importance Sampling method, to avoid the problem known as degeneration of particles. Such filter is known as Sampling Importance Re-sampling (SIR) Filter. In 2008, Orlande et al [20] presented an application of the SIR Filter to a linear heat conduction problem.

In order to overcome some difficulties of SIR filter, Pitt and Shepard [21] introduced the Auxiliary Particle Filter (APF). In 2006, Del Moral et al [16] presented also an alternative to improve the SIR Filter, named Sequential Monte Carlo Samplers, introducing a method for the evolution of the particles and also an artificial delayed kernel.

Another well-known filter is the Combined Parameter and State Estimation in Simulation Based Filtering, proposed by Liu and West [22], which uses a combination of the artificial evolution method (where the problem related with the lost of information is avoided) and the smoothness kernel proposed by West [23], which improve the choice of the particles. In 2007, Sisson and Fran [24] presented a new filter technique, where the Sequential Monte Carlo Method was coupled with the Approximate Bayesian Computation, named Sequential Monte Carlo without Likelihoods.

In this paper we applied the Auxiliary Sampling Importance Resampling (ASIR) algorithm to the estimate of an unknown heat flux at a top wall of a square cavity undergoing a natural convection process. Two different heat fluxes profiles were estimated with very good results. Also, the number of particles as well as the frequency of the measurements were analyzed.

PHYSICAL PROBLEM

The physical problem under picture in this paper involves the transient laminar natural convection of a fluid inside a two-dimensional square cavity. The fluid is initially at rest and at the uniform temperature T_c. At time zero, the bottom and top surfaces are subjected to time-dependent heat fluxes $q_1(t)$ and $q_2(t)$, respectively. The left and right surfaces are subjected to constant temperatures T_c and T_h, respectively. The fluid properties are assumed constant, except for the density in the buoyancy term, where we consider Boussinesq's approximation valid.

The mathematical formulation for this physical problem can be written in vector form in terms of the following conservation equation in the generalized Cartesian coordinates:

$$\frac{\partial(\rho\boldsymbol{\varphi})}{\partial t}+\frac{\partial(u\rho\boldsymbol{\varphi})}{\partial x}+\frac{\partial(v\rho\boldsymbol{\varphi})}{\partial y}=\nabla\bullet\left(\boldsymbol{\Gamma}^{\phi}\nabla\boldsymbol{\varphi}\right)+\mathbf{S} \tag{1}$$

The general conservation variable, as well as the diffusion coefficient and the source-term for the mass, momentum and energy conservation equations, are given in vector form respectively as:

$$\boldsymbol{\varphi}=\begin{bmatrix}1\\ u(x,y,t)\\ v(x,y,t)\\ T(x,y,t)\end{bmatrix}\qquad \boldsymbol{\Gamma}^{\phi}=\begin{bmatrix}0&0&0&0\\0&\mu&0&0\\0&0&\mu&0\\0&0&0&\dfrac{K}{C_P}\end{bmatrix} \tag{2.a,b}$$

$$\mathbf{S}^{\phi}=\begin{bmatrix}0\\ -\dfrac{\partial P(x,y,t)}{\partial x}\\ -\dfrac{\partial P(x,y,t)}{\partial y}-\rho g\left\{1-\beta\left[T(x,y,t)-T_{ref}\right]\right\}\\ 0\end{bmatrix} \tag{2.c}$$

We note in the Eq. (2.c) that the positive y-axis in the physical domain is supposed to be aligned with the opposite direction of the gravitational acceleration vector. These equations are solved, subjected to the following boundary and initial conditions

$T=T_c$	at x=1, 1<y<H, for t>0	(3.a)
$T=T_h$	at x=W, 1<y<H, for t>0	(3.b)
$u=v=0$	at x=1 and x=W, 1<y<H, for t> 0	(3.c)
$u=v=0$	at y=1 and y=H, 1<x<W, for t> 0	(3.d)
$K\dfrac{\partial T}{\partial y}=-q_1(t)$	at y=1, 1<x<W, for t> 0	(3.e)

$$K\frac{\partial T}{\partial y}=q_2(t) \qquad \text{at } y=H,\ 1<x<W,\ \text{for } t>0 \tag{3.f}$$

$$u=v=0 \qquad \text{for } t=0 \text{ in the region} \tag{3.g}$$

$$T=T_c \qquad \text{for } t=0 \text{ in the region} \tag{3.h}$$

The SIMPLEC method [25] was used to solve velocity-pressure coupling problem. The WUDS interpolation scheme [26] was used to obtain the values of u, v and T as well as their derivatives at the interfaces of each control volume. The resulting linear system was solved by the GMRES method [27].

INVERSE PROBLEM

The solution of the inverse problem within the Bayesian framework is recast in the form of statistical inference from the *posterior probability density*, which is the model for the conditional probability distribution of the unknown parameters given the measurements. The measurement model incorporating the related uncertainties is called the *likelihood*, that is, the conditional probability of the measurements given the unknown parameters. By assuming that the measurement errors are Gaussian random variables, with zero means and known covariance matrix $\mathbf{W}$ and that the measurement errors are additive and independent of the parameters $\mathbf{P}$, the *likelihood function* can be expressed as [1,3,28-32]

$$\pi(\mathbf{Y}|\mathbf{P})=(2\pi)^{-D/2}|\mathbf{W}|^{-1/2}\exp\left\{-\frac{1}{2}[\mathbf{Y}-\mathbf{T}(\mathbf{P})]^T\mathbf{W}^{-1}[\mathbf{Y}-\mathbf{T}(\mathbf{P})]\right\} \tag{4}$$

where $\mathbf{Y}$ are the measurements and $\mathbf{T}(\mathbf{P})$ is the solution of the direct (forward) problem. Such solution is obtained from the mathematical formulation of the heat transfer problem under analysis with known $\mathbf{P}$.

The model for the unknowns that reflects all the uncertainty of the parameters without the information conveyed by the measurements, is called the *prior* model [1,3,28-32].

The formal mechanism to combine the new information (measurements) with the previously available information (prior) is known as the Bayes' theorem [1,3,28-32]. Therefore, the term Bayesian is often used to describe the statistical inversion approach, which is based on the following principles [1]: 1. All variables included in the model are modeled as random variables; 2. The randomness describes the degree of information concerning their realizations; 3. The degree of information concerning these values is coded in probability distributions; and 4. The solution of the inverse problem is the posterior probability distribution, from which distribution point estimates and other statistics are computed. Therefore, this approach relies fundamentally on the principles of the Bayesian statistics to obtain the solution of inverse problems. Recent works on the application of Bayesian techniques to inverse heat transfer problems include references [33-46].

Bayes' theorem is stated as [1,3,28-32]:

$$\pi_{posterior}(\mathbf{P})=\pi(\mathbf{P}|\mathbf{Y})=\frac{\pi(\mathbf{P})\pi(\mathbf{Y}|\mathbf{P})}{\pi(\mathbf{Y})} \tag{5}$$

where $\pi_{posterior}(\mathbf{P})$ is the posterior probability density, $\pi(\mathbf{P})$ is the prior density, $\pi(\mathbf{Y}|\mathbf{P})$ is the likelihood function and $\pi(\mathbf{Y})$ is the marginal probability density of the measurements, which plays the role of a normalizing constant.

State Estimation

State estimation problems, also designated as nonstationary inverse problems [1], are of great interest in innumerable practical applications. In such kinds of problems, the available measured data is used together with prior knowledge about the physical phenomena and the measuring devices, in order to sequentially produce estimates of the desired dynamic variables. This is accomplished in such a manner that the error is minimized statistically [2].

Consider a model for the evolution of the state variables $\mathbf{x}$ in the form

$$\mathbf{x}_k=\mathbf{f}_k(\mathbf{x}_{k-1},\mathbf{v}_k) \tag{6}$$

where $\mathbf{f}$ is, in the general case, a non-linear function of $\mathbf{x}$ and of the state noise or uncertainty vector given by $\mathbf{v}_k\in\mathbf{R}^{\mathbf{n}}$. The vector $\mathbf{x}_k\in\mathbf{R}^{\mathbf{n}}$ is called the state vector and contains the variables to be dynamically estimated. This vector advances in time in accordance with the *state evolution model* (6). The subscript k =1, 2, 3, …, denotes a time instant t_k in a dynamic problem.

The observation model describes the dependence between the state variable $\mathbf{x}$ to be estimated and the measurements $\mathbf{z}$ through the general, possibly non-linear, function $\mathbf{h}$. This can be represented by

$$\mathbf{z}_k=\mathbf{h}_k(\mathbf{x}_k,\mathbf{n}_k) \tag{7}$$

where $\mathbf{z}_k\in\mathbf{R}^{\mathbf{nz}}$ are available at times t_k, k=1, 2, 3,…. Eq. (7) is referred to as the *observation/measurement model*. The vector $\mathbf{n}_k\in\mathbf{R}^{\mathbf{nz}}$ represents the measurement noise or uncertainty.

As per equations (6) and (7), the *evolution and observation models* are based on the following assumptions [1,2,4-9,46]:

(a) The sequence $\mathbf{x}_k$ for k=1, 2, 3, …, is a Markovian process, that is,

$$\pi(\mathbf{x}_k|\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1})=\pi(\mathbf{x}_k|\mathbf{x}_{k-1}) \tag{8.a}$$

(b) The sequence $\mathbf{z}_k$ for k=1, 2, 3, …, is a Markovian process with respect to the history of $\mathbf{x}_k$, that is,

$$\pi(\mathbf{z}_k|\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_k)=\pi(\mathbf{z}_k|\mathbf{x}_k) \tag{8.b}$$

(c) The sequence $\mathbf{x}_k$ depends on the past observations only through its own history, that is,

$$\pi(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_{k-1})=\pi(\mathbf{x}_k|\mathbf{x}_{k-1}) \tag{8.c}$$

where $\pi(\mathbf{a}|\mathbf{b})$ denotes the conditional probability of **a** when **b** is given.

For the state and observation noises, the following assumptions are made [1,2,4-9,46]:

(a) For $i \neq j$, the noise vectors $\mathbf{v}_i$ and $\mathbf{v}_j$, as well as $\mathbf{n}_i$ and $\mathbf{n}_i$, are mutually independent and also mutually independent of the initial state $\mathbf{x}_0$.
(b) The noise vectors $\mathbf{v}_i$ and $\mathbf{n}_j$ are mutually independent for all i and j.

Different problems can be considered for the evolution-observation models described above, such as [1,2,4-9,46]:

(i) The *prediction problem*, when the objective is to obtain $\pi(\mathbf{x}_k|\mathbf{z}_{1:k-1})$;
(ii) The *filtering problem*, when the objective is to obtain $\pi(\mathbf{x}_k|\mathbf{z}_{1:k})$;
(iii) The *fixed-lag smoothing problem*, when the objective is to obtain $\pi(\mathbf{x}_k|\mathbf{z}_{1:k+p})$, where $p \geq 1$ is the fixed lag.
(iv) The *whole-domain smoothing problem*, when the objective is to obtain $\pi(\mathbf{x}_k|\mathbf{z}_{1:K})$, where $\mathbf{z}_{1:K}=\{\mathbf{z}_i, i=1,...,K\}$ is the complete set of measurements.

We consider here the filtering problem. By assuming that $\pi(\mathbf{x}_0|\mathbf{z}_0)= \pi(\mathbf{x}_0)$ is available, the posterior probability density $\pi(\mathbf{x}_k|\mathbf{z}_{1:k})$ is then obtained with Bayesian filters in two steps [1,2,4-9]: *prediction and update*, as illustrated in figure 1.

The most widely known Bayesian filter method is the Kalman filter [1,2,4-17,20,46-48]. However, the application of the Kalman filter is limited to linear models with additive Gaussian noises. Extensions of the Kalman filter were developed in the past for less restrictive cases by using linearization techniques. Similarly, Monte Carlo methods have been developed in order to represent the posterior density in terms of random samples and associated weights. Such Monte Carlo methods, usually denoted as particle filters among other designations found in the literature, do not require the restrictive hypotheses of the Kalman filter. Hence, particle filters can be applied to non-linear models with non-Gaussian errors [1,2,4-17,20,47,48].

The main idea in the particle filter is to represent the required posterior density function by a set of random samples with associated weights and to compute the estimates based on these samples and weights [1,8-17,20,47,48]. Let $\{\mathbf{x}^i_{0:k},\ i=0,...,N\}$ be the particles with associated weights $\{\mathbf{w}^i_k,\ i=0,...,N\}$ and $\mathbf{x}_{0:k}=\{\mathbf{x}_j,\ j=0,...,k\}$ be the set of all states up to t_k, where N is the number of particles. The weights are normalized, so that $\Sigma_i\, w^i_k=1$. Then, the posterior density at t_k can be discretely approximated by:

$$\pi(\mathbf{x}_{0:\mathrm{k}}|\mathbf{z}_{1:\mathrm{k}-1}) \approx \sum_{i=1}^{I} w_k^i\, \delta(\mathbf{x}_{0:\mathrm{k}} - \mathbf{x}^{\mathrm{i}}_{0:\mathrm{k}}) \qquad (9)$$

where $\delta(.)$ is the Dirac delta function. By taking hypotheses (8.a-c) into account, the posterior density in Eq. (9) can be written as $\pi(\mathbf{x}_k|\mathbf{z}_{1:k-1}) \approx \Sigma_i\, w^i_k\, \delta(\mathbf{x}_k-\mathbf{x}^i_k)$.

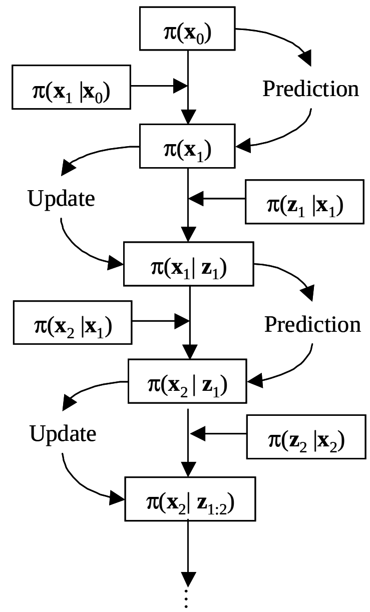


Figure 1. Prediction and update steps [1]

A common problem with the Particle Filter method is the degeneracy phenomenon, where after a few states all but one particle may have negligible weight. The degeneracy implies that a large computational effort is devoted to updating particles whose contribution to the approximation of the posterior density function is almost zero. This problem can be overcome by increasing the number of particles, or more efficiently by appropriately selecting the importance density as the prior density $\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$. In addition, the use of the resampling technique is recommended to avoid the degeneracy of the particles [1,8-17,20,47,48].

Resampling generally involves a mapping of the random measure $\{\mathbf{x}^i_k, w^i_k\}$ into a random measure $\{\mathbf{x}^{i^*}_k, N^{-1}\}$ with uniform weights. It can be performed if the number of effective particles with large weights falls below a certain threshold number. Alternatively, resampling can also be applied indistinctively at every instant t_k, as in the *Auxiliary Sampling Importance Resampling* (ASIR) algorithm [8,9]. This algorithm can be summarized in the steps presented in Table 1, as applied to the system evolution from t_{k-1} to t_k.

Although the resampling step reduces the effects of the degeneracy problem, it may lead to a loss of diversity and the resultant sample can contain many repeated particles. This problem, known as sample impoverishment, can be severe in the case of small evolution model noise. In this case, all particles collapse to a single particle within few instants t_k. Another drawback of the particle filter is related to the large computational cost due to the Monte Carlo method, which may limit its application only to fast computing problems. Different algorithms for the implementation of the particle filter can be found in [48], including those that permit the simultaneous estimation of constant parameters appearing in the model and the transient states.

Table 1 – ASIR Algorithm [8,9]

Step 1
For i=1,...,N draw new particles $\mathbf{x}_k^i$ from the prior density $\pi(\mathbf{x}_k\vert\mathbf{x}^i_{k-1})$ and then calculate some characterization of $\mathbf{x}_k$, given $\mathbf{x}^i_{k-1}$, as for example the mean μ^i_k=E[$\mathbf{x}_k\vert\mathbf{x}^i_{k-1}$]. Then use the likelihood density to calculate the correspondent weights $w^i_k=\pi(\mathbf{z}_k\vert\mu^i_k)w^i_{k-1}$
Step 2
Calculate the total weight $t=\Sigma_i\ w^i_k$ and then normalize the particle weights, that is, for i=1,...,N let $w^i_k = t^{-1}\ w^i_k$
Step 3
Resample the particles as follows :
Construct the cumulative sum of weights (CSW) by computing $c_i=c_{i-1}+w^i_k$ for i=1,...,N, with c_0=0
Let i=1and draw a starting point u_1 from the uniform distribution U[0,N^{-1}]
For j=1,...,N
Move along the CSW by making $u_j=u_1+N^{-1}(j-1)$
While $u_j>c_i$ make $i=i+1$ Assign sample $x^j_k=x^i_k$ Assign sample $w^j_k=N^{-1}$ Assign parent $i^j=i$
Step 4
For j=1,...,N draw particles $\mathbf{x}_k^j$ from the prior density $\pi(\mathbf{x}_k\vert\mathbf{x}^{ij}_{k-1})$, using the parent i^j, and then use the likelihood density to calculate the correspondent weights $w^j_k=\pi(\mathbf{z}_k\vert\mathbf{x}^j_k)\ /\ \pi(\mathbf{z}_k\vert\mu^{ij}_k)$
Step 5
Calculate the total weight $t=\Sigma_j\ w^j_k$ and then normalize the particle weights, that is, for j=1,...,N let $w^j_k = t^{-1}\ w^j_k$

According to [8], the advantage of the ASIR filter over the *Sampling Importance Resampling* (SIR) algorithm is that it naturally generates points from the sample at k-1, which, conditioned on the current measurement, are most likely to be close to the true state. Yet as described in [8], ASIR can be viewed as resampling at the previous time step, based on some point estimates μ^i_k that characterize $\pi(\mathbf{x}_k\vert\mathbf{x}^i_{k-1})$. The use of ASIR is limited to small process noise. For a large process noise, a single point μ^i_k is not able to characterize $\pi(\mathbf{x}_k\vert\mathbf{x}^i_{k-1})$.

RESULTS AND DISCUSSIONS

In this paper we applied the ASIR filter to estimate a time-varying heat flux applied to the top wall of a square cavity filled with air (ρ=1.19 kg/m^3; K=0.02624 W/(m.K); C_p=1035.0222 J/(kg.K); μ=1.8x10^{-5} kg/(m.s); β=0.00341 K^{-1}). The bottom wall of the cavity was kept thermally insulated and the left and right walls were subjected to constant temperatures equal to 2^{o}C and 12^{o}C, respectively. The width and height of the cavity were equal to 0.045841 m, which resulted in a Rayleigh (Ra) number equals to 10^5, where

$$Ra = \frac{\rho^2 C_p g\beta(T_h - T_c)W^3}{\mu K} \tag{10}$$

The state estimate problem consists thus in predict the behavior of the state variable $q_2(t)$ at the top wall of the cavity. However, since the heat flux affects the temperature field through the energy equation and also the mass and momentum equations through the buoyancy source term, Eqs. (1)-(3), the state vector $\mathbf{x}$ appearing in Eq. (6) is composed of the heat flux $q_2(t)$, plus all velocity and temperature fields inside the cavity. Due to tremendous amount of computational resources involved for the solution of this problem, we used a very course finite volume grid (11x11 volumes) to demonstrate the feasibility of the method. In future works, a finer grid shall be used. The total number of state variables are thus: 1 for the heat flux $q_2(t)$, 11x11 for the u component of the velocity field, 11x11 for the v component of the velocity field, and 11x11 for the temperature T, resulting in 364 state variables.

In the framework of the particle filters, two auxiliary models are needed: (i) an evolution model, given by Eq. (6), and (ii) a observation model, given by Eq. (7). For the evolution model, we used two sub-models: (i.i) a evolution model for the velocity and temperature fields, given by the solution of mass, momentum and energy equations, Eqs. (1)-(3), where the state noise $\mathbf{v}$, appearing on Eq. (6), was supposed to be equal to 1% of the state variable

$$\mathbf{x}_k = \mathbf{x}_k + \boldsymbol{\sigma}\varepsilon \Rightarrow \begin{array}{l} u(t) = u(t)[1+0.01\varepsilon] \\ v(t) = v(t)[1+0.01\varepsilon] \\ T(t) = T(t)[1+0.01\varepsilon] \end{array} \tag{11}$$

where ε is a random variable with Gaussian distribution and zero mean; and (i,ii) an evolution model for the heat flux $q_2(t)$ which was taken as a random walk model

$$q_2(t) = q_2(t-1) + \sigma_q \varepsilon \tag{12}$$

where σ_q varied automatically between 10% and 100% of the value of $q_2(t$-1).

For the observation model, we used simulated temperature measurements, where an experimental error with standard deviation equals to 1% of the local value of the temperature was used. Such measurements were taken at the top and bottom walls of the cavity, in 11points equally spaced at each wall.

Two different profiles were tried for the heat flux $q_2(t)$. Also, two different number of particles and two frequencies of observation were analyzed. Table 2 summarizes the test cases analyzed. For all test cases, the initial guess for the heat flux was equal to zero with a unity standard deviation.

Figure 2 shows the estimated values of $q_2(t)$ with the linear profile (test cases 1-4). The average values at each time are shown by the symbols with error bars corresponding to a 99% confidence interval. For the case with 10 particles and a measurement frequency equals to 1 Hz, there was an initial delay of almost 200 seconds in the estimation of such heat flux. However, when the number of particles was increased to 100, with the same measurement rate of 1 Hz, the particle filter was able to fully recover the unknown heat. Also, when more particles were used, the error bars were more uniform during all time period analyzed, whereas for 10 particles there was a fluctuation in the error bars as can be seen in Fig. 2. One interesting result has to do with the frequency of measurements. From Fig. 2 one can see that when such frequency was increased from 1 Hz to 10 Hz the results became worst, with a large fluctuation of the average heat flux and larger values of the error bars. WHY??? IT WOULD BE POSSIBLE THAT LOW FREQUENCIES SMOOTH THE ERRORS OF THE MEASUREMENTS, WHILE HIGH FREQUENCIES MAKE THE FILTER MORE OSCILATORY???

Table 2 - Test cases analyzed

Case	Heat flux	Particles	Frequency
1	$q_2(t)=0.01\,t$ [W/m^2]	10	1 Hz
2			10 Hz
3		100	1 Hz
4			10 Hz
5	$q_2(t)=0$ W/m^2 for t<500s $q_2(t)=5$ W/m^2 for t>500s	10	1 Hz
6			10 Hz
7		100	1 Hz
8			10 Hz

Figure 3 shows the real and recovered temperature profiles, while figure 4 shows the real and recovered streamlines, where one can notice the excellent estimate of the temperature and velocity fields. From figure 4, one can see that the streamlines at 100 seconds for the test case with 10 particles and a measurement frequency equals to 1 Hz were not very well captured, since the filter presented an initial delay of approximately 200 seconds to estimate such heat flux as discussed previously.

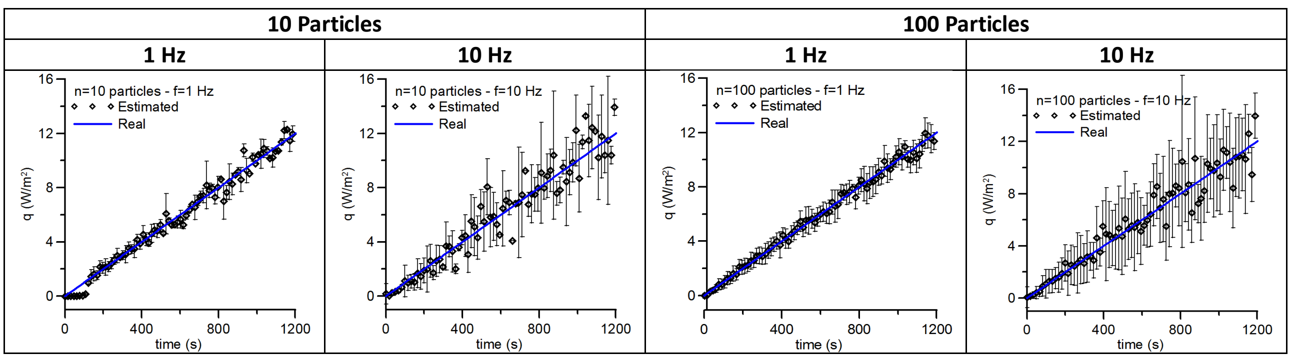
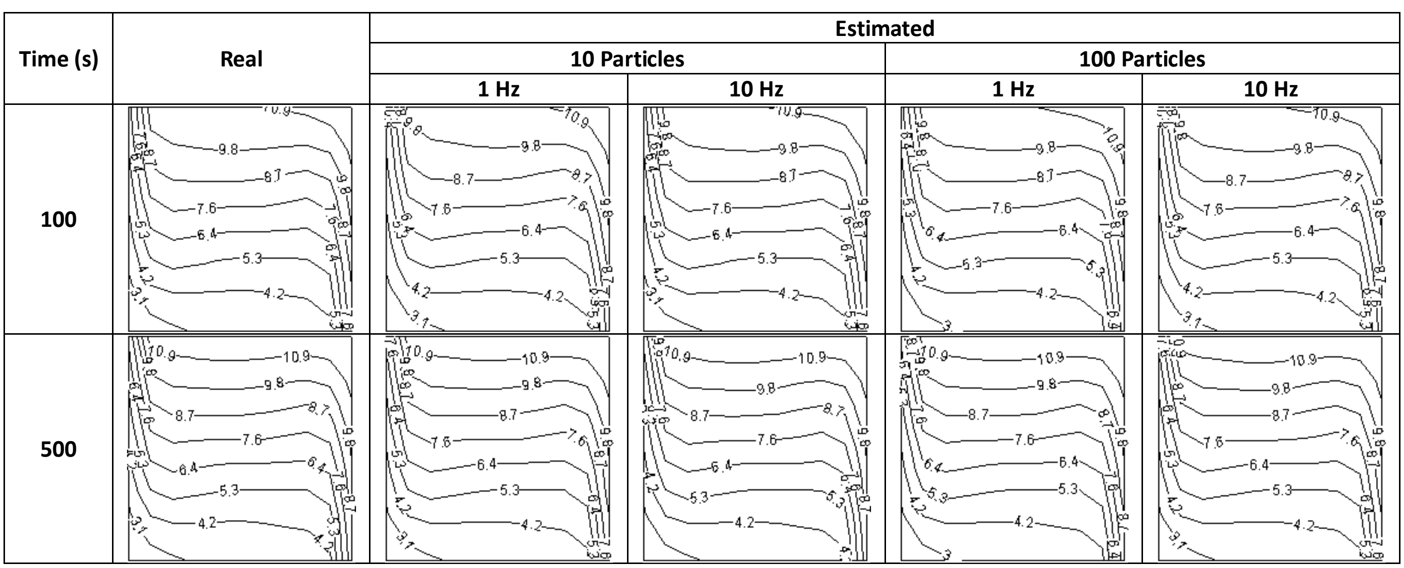


Figure 2 - Estimated heat flux with the linear profile



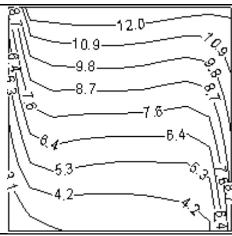
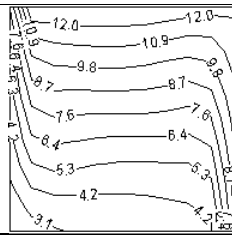
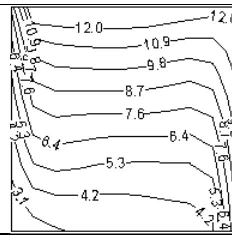
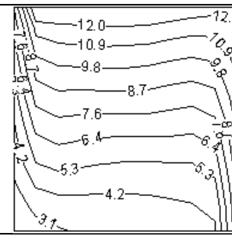
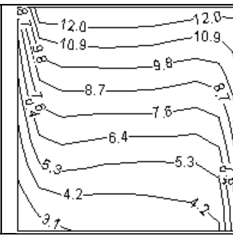
1000					

Figure 3 - Real and estimated temperature profiles with the linear heat flux profile

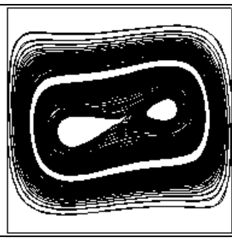
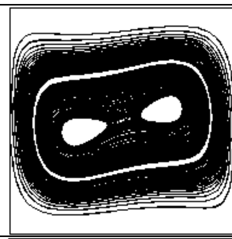
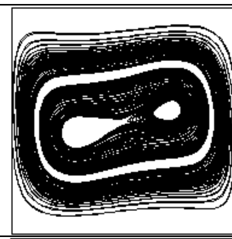
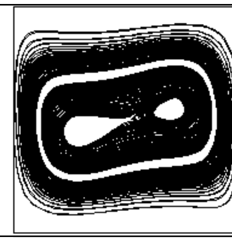
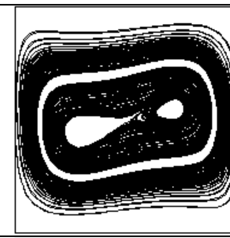
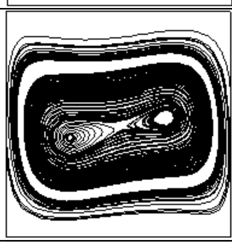
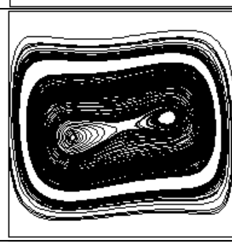
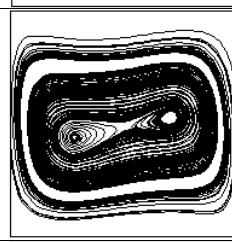
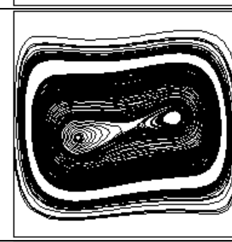
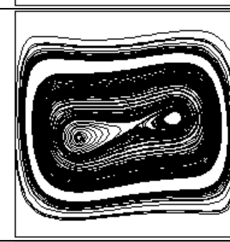
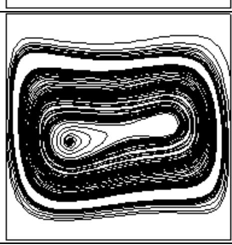
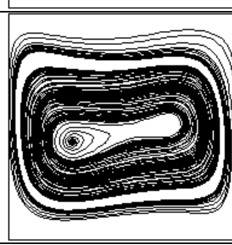
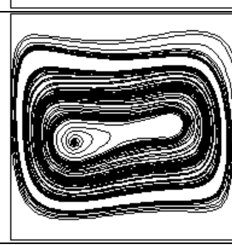
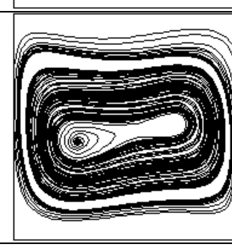
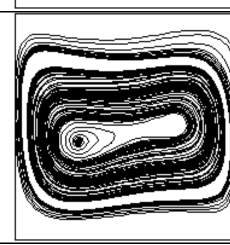
Time (s)	**Real**	**Estimated**			
		10 Particles		**100 Particles**	
		1 Hz	**10 Hz**	**1 Hz**	**10 Hz**
100					
500					
1000					

Figure 4 - Real and estimated streamlines with the linear heat flux profile

Figure 5 shows the estimated heat flux with the step profile (test cases 5-8) obtained by the particle filter methodology. Again, for 10 particles with a measurement rate equals to 1 Hz there was a delay to estimate the discontinuity at 500 seconds. When the number or particles was increased to 100, such delay was decreased. Also, the spread of the average value of the heat flux was less pronounced when more particles (100 instead of 10) were used. Once again, the error bars increased when the frequency of the measurement increased from 1 Hz to 10 Hz. WHY?

Figure 6 shows the real and recovered temperature profiles, while figure 7 shows the real and recovered streamlines, where one can notice the excellent estimate of the temperature and velocity fields.

10 Particles		**100 Particles**	
1 Hz	**10 Hz**	**1 Hz**	**10 Hz**

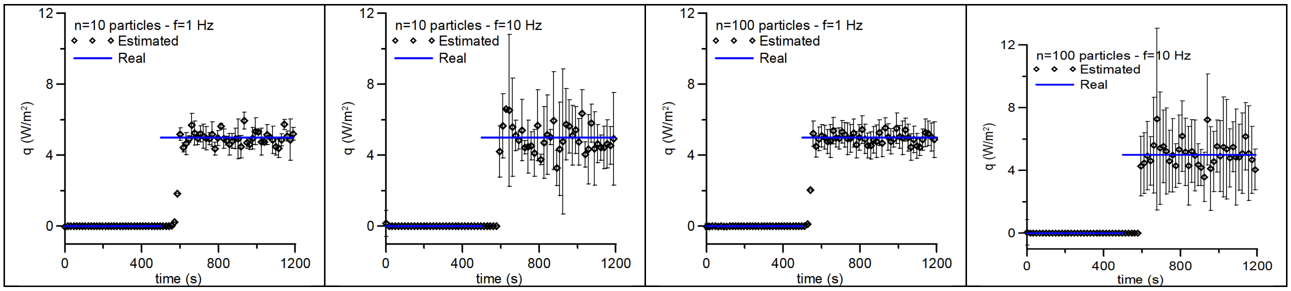


Figure 5 - Estimated heat flux with the step profile

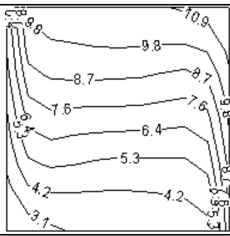
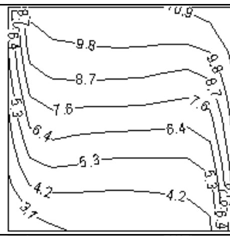
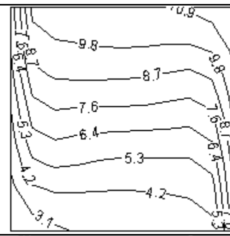
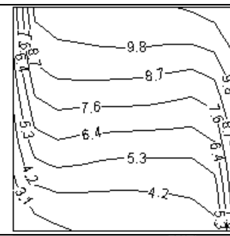
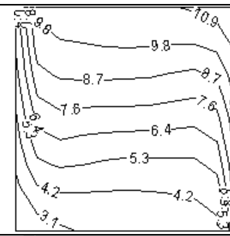
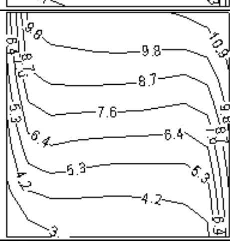
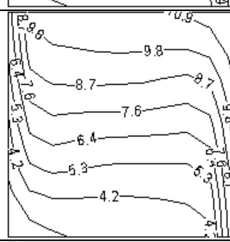
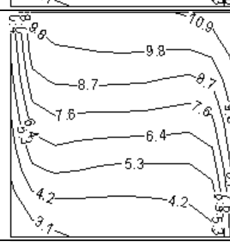
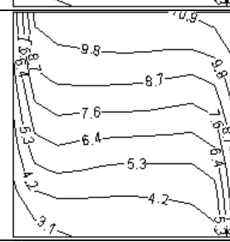
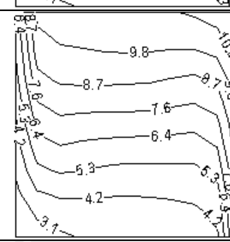
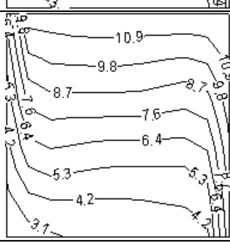
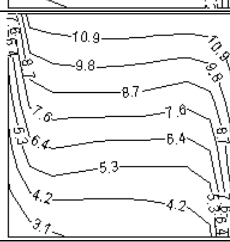
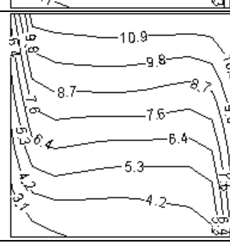
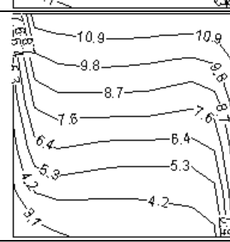
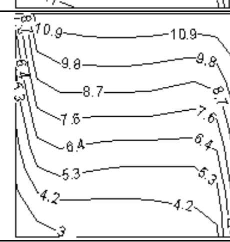
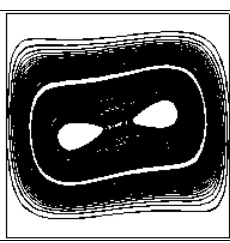
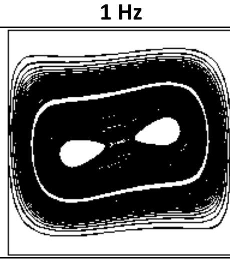
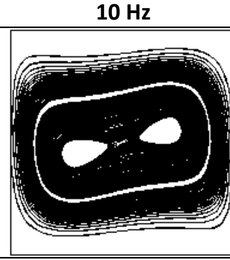
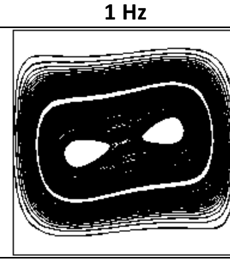
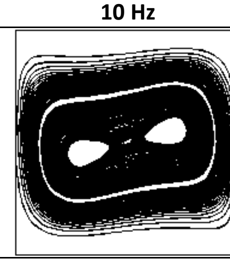
Time (s)	Real	Estimated			
		10 Particles		100 Particles	
		1 Hz	10 Hz	1 Hz	10 Hz
100					
500					
1000					

Figure 6 - Real and estimated temperature profiles with the step heat flux profile

Time (s)	Real	Estimated			
		10 Particles		100 Particles	
		1 Hz	10 Hz	1 Hz	10 Hz
100					

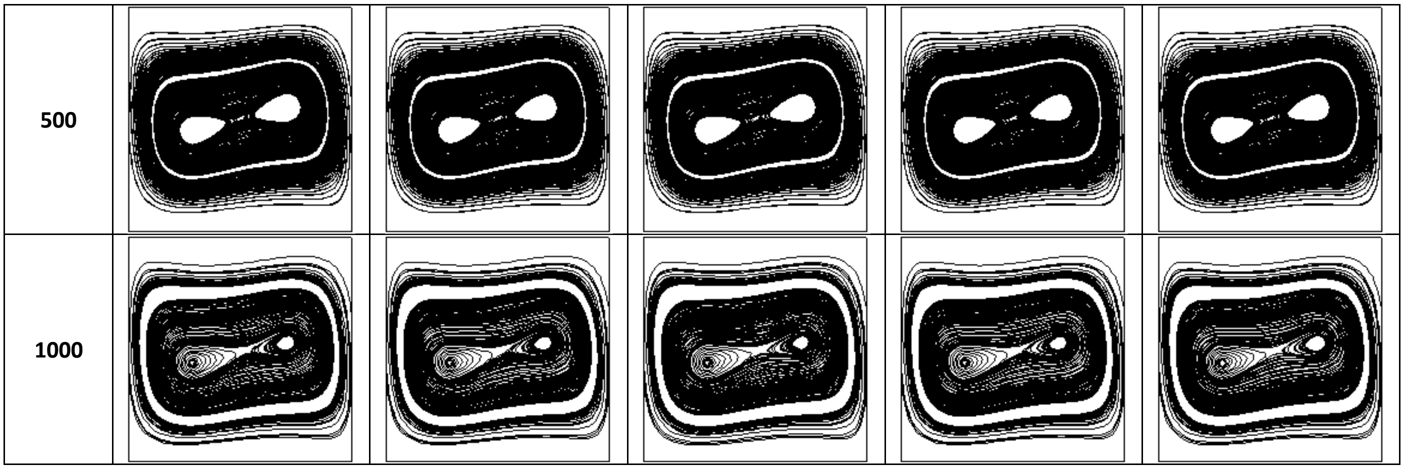


Figure 7 - Real and estimated streamlines with the step heat flux profile

CONCLUSIONS

In this paper we applied the Auxiliary Sampling Importance Resampling (ASIR) algorithm to the estimate of an unknown heat flux at a top wall of a square cavity undergoing a natural convection process. Two different heat fluxes profiles were estimated with very good results. Also, the number of particles as well as the frequency of the measurements was analyzed, showing that as the frequency decreases, the results improve, with lower error bars. Also, when the number of particles increases, the results become less spread around the exact value of the unknown heat flux.

ACKNOWLEDGEMENTS

The authors would like to thank the Brazilian agencies for the fostering of science, *Conselho Nacional de Desenvolvimento Científico e Tecnológico* (CNPq), *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior* (CAPES) and *Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro* (FAPERJ), for the financial support for this work.

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