A Fourier Series Method for Solving Ordinary Differential Equations with Non-Constant Coefficients Arising in Inverse Shape Design

Daniel P. Baker[1] and George S. Dulikravich[2]
[1]P.O. Box 124, Lemont, PA 16851
Email: baker.daniel@verizon.net
[2]Mechanical and Materials Engineering Department
Florida International University, Miami, FL 33174
Email: dulikrav@fiu.edu

Abstract

An analytical method of integrating ordinary differential equations with non-constant coefficients arising from an elastic membrane concept for inverse shape design is presented utilizing Fourier series formulation. The non-homogeneous ordinary differential equation with non-constant coefficients mimics forced oscillations of a system of mass-damper-spring elements linked in parallel where coefficients of mass, damper and spring are non-constant. This elastic membrane concept for inverse shape design requires knowledge only of the surface field variables distribution on the body to perform a shape update. Thus, it can be implemented without modifying an existing field analysis code. The proposed formulation allows each segment of an evolving shape to move at its own optimal speed thus potentially significantly reducing the required number of shape updates until it matches the specified surface field data.

1. Introduction

The elastic membrane approach to aerodynamic inverse shape design was first proposed by Garabedian and McFadden [1]. This concept treats the surface of an aerodynamic body as a membrane that deforms under aerodynamic loads until it achieves specified (target) surface pressure distribution. The original model [1] for the evolution of an airfoil shape to create a specified (target) surface pressure distribution was given by a simple mass-damper-spring equation for forced vibrations

$$\beta_0 \Delta n + \beta_1 \frac{d\Delta n}{dx} + \beta_2 \frac{d^2 \Delta n}{dx^2} = C_p^{\text{target}} - C_p^{\text{actual}} \tag{1}$$

Here, Δn's are defined as shape corrections along outward normal vectors, and $\beta_{0\text{-}2}$ are user supplied constants coefficients that control the rate of convergence of the shape evolution process, while C_p^{target} and C_p^{actual} are the specified (target) and the actual (computed) local surface pressure coefficients, respectively. This technique was modified by Malone *et al.* [2], giving

$$\beta_0 \Delta y + \beta_1 \frac{d\Delta y}{dx} + \beta_2 \frac{d^2 \Delta y}{dx^2} = C_p^{\text{target}} - C_p^{\text{actual}} \tag{2}$$

so that shape modifications are in the y-direction only, thus preventing the chord length from changing. Equation (2) is traditionally solved for Δy shape corrections using a finite difference approach by discretizing along the airfoil contour. This approach has slow convergence especially with the flow-field analysis codes of increasing non-linearity [3-6] because of the truncation errors resulting from numerical differentiation. The iterative process of evolving solution of equation (2) has been significantly accelerated by using Fourier series formulation for a *de facto* analytical integration of equation (2) as presented by Baker and Dulikravich [3-7]. However, even this can be improved upon if it could be possible to have each segment of a body surface move with its own values of coefficients of the mass, damper and spring, that is, by allowing these coefficients to be arbitrary functions of the x-coordinate.

This paper offers an attempt to derive an analytical framework for integrating such ordinary differential equations with non-constant coefficients by utilizing Fourier series.

2. Analytical Formulation

Let us represent two functions, $Y_u(x)$ and $Y_k(x)$, in terms of complete Fourier series so that

$$Y_u = \sum_{n=0}^{n_{max}} \left[a_n \cos(nx) + b_n \sin(nx)\right] \tag{3}$$

$$Y_k = \sum_{n=0}^{n_{max}} \left[e_n \cos(nx) + f_n \sin(nx)\right] \tag{4}$$

These two Fourier series (with the same number of terms) multiply together to make another Fourier series that will have an equal number of terms.

$$\begin{aligned} Y_k Y_u &= \sum_{n=0}^{n_{max}} [a_n \cos(nx) + b_n \sin(nx)] \sum_{m=0}^{n_{max}} [e_m \cos(mx) + f_m \sin(mx)] = \\ &\sum_{n=0}^{n_{max}} \sum_{m=0}^{n_{max}} \{a_n [e_m \cos(mx) + f_m \sin(mx)] \cos(nx) + b_n [e_m \cos(mx) + f_m \sin(mx)] \sin(nx)\} \end{aligned} \tag{5}$$

The following four trigonometric identities will simplify this.

$$\begin{aligned} \cos(nx)\cos(mx) &= \frac{1}{2}[\cos(n+m)x + \cos(n-m)x] \\ \cos(nx)\sin(mx) &= \frac{1}{2}[\sin(n+m)x - \sin(n-m)x] \\ \sin(nx)\cos(mx) &= \frac{1}{2}[\sin(n+m)x + \sin(n-m)x] \\ \sin(nx)\sin(mx) &= \frac{1}{2}[-\cos(n+m)x + \cos(n-m)x] \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} Y_k Y_u &= \frac{1}{2} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n_{max}} [(e_m a_n + f_m b_n) \cos(n-m)x + (e_m b_n - f_m a_n) \sin(n-m)x] \\ &+ \sum_{n=0}^{n_{max}} \sum_{m=0}^{n_{max}} [(e_m a_n - f_m b_n) \cos(n+m)x + (e_m b_n + f_m a_n) \sin(n+m)x] \end{aligned} \tag{7}$$

or

$$\begin{aligned} Y_k Y_u &= \frac{1}{2} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n} [(e_m a_n + f_m b_n) \cos(n-m)x + (e_m b_n - f_m a_n) \sin(n-m)x] \\ &+ \frac{1}{2} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n_{max}-n} [(e_m a_n - f_m b_n) \cos(n+m)x + (e_m b_n + f_m a_n) \sin(n+m)x] \\ &\frac{1}{2} \sum_{n=0}^{n_{max}} \sum_{m=n+1}^{n_{max}} [(e_m a_n + f_m b_n) \cos(n-m)x + (e_m b_n - f_m a_n) \sin(n-m)x] \\ &+ \frac{1}{2} \sum_{n=0}^{n_{max}} \sum_{m=n_{max}-n+1}^{n_{max}} [(e_m a_n - f_m b_n) \cos(n+m)x + (e_m b_n + f_m a_n) \sin(n+m)x] \end{aligned} \tag{8}$$

The first two lines of the above sum produce harmonics with wavenumber inclusively between zero and nmax. The third line produces wavenumbers less than zero. The fourth line produces wavenumbers greater than n_{max}. The third line is corrected using the following identities.

$$\begin{aligned} \cos(-A) &= \cos(A) \\ \sin(-A) &= -\sin(A) \end{aligned} \tag{9}$$

This results in

$$\begin{aligned} Y_k Y_u &= \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=0}^{n}[(e_m a_n + f_m b_n)\cos(n-m)x + (e_m b_n - f_m a_n)\sin(n-m)x] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=0}^{n_{max}-n}[(e_m a_n - f_m b_n)\cos(n+m)x + (e_m b_n + f_m a_n)\sin(n+m)x] \\ &\frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=n+1}^{n_{max}}[(e_m a_n + f_m b_n)\cos(m-n)x - (e_m b_n - f_m a_n)\sin(m-n)x] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=n_{max}-n+1}^{n_{max}}[(e_m a_n - f_m b_n)\cos(n+m)x + (e_m b_n + f_m a_n)\sin(n+m)x] \end{aligned} \tag{10}$$

Wavenumbers greater than n_{max} will be aliased back into the available wavenumber spectrum by an FFT, thus modifying the fourth line as follows

$$\begin{aligned} Y_k Y_u &= \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=0}^{n}[(e_m a_n + f_m b_n)\cos(n-m)x + (e_m b_n - f_m a_n)\sin(n-m)x] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=0}^{n_{max}-n}[(e_m a_n - f_m b_n)\cos(n+m)x + (e_m b_n + f_m a_n)\sin(n+m)x] \\ &\frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=n+1}^{n_{max}}[(e_m a_n + f_m b_n)\cos(m-n)x - (e_m b_n - f_m a_n)\sin(m-n)x] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{m=n_{max}-n+1}^{n_{max}}[(e_m a_n - f_m b_n)\cos(2n_{max}-n-m)x - (e_m b_n + f_m a_n)\sin(2n_{max}-n-m)x] \end{aligned} \tag{11}$$

This expression can be rewritten in a more convenient form by changing the subscripts so that m = n – t in the first line, m = t – n in the second line, m = t + n in the third line, and m = 2nmax – n – t in the fourth.

$$\begin{aligned} Y_k Y_u &= \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{t=0}^{n}[(e_{n-t} a_n + f_{n-t} b_n)\cos(tx) + (e_{n-t} b_n - f_{n-t} a_n)\sin(tx)] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{t=n}^{n_{max}}[(e_{t-n} a_n - f_{t-n} b_n)\cos(tx) + (e_{t-n} b_n + f_{t-n} a_n)\sin(tx)] \\ &\frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{t=1}^{n_{max}-n}[(e_{t+n} a_n + f_{t+n} b_n)\cos(tx) - (e_{t+n} b_n - f_{t+n} a_n)\sin(tx)] \\ &+ \frac{1}{2}\sum_{n=0}^{n_{max}}\sum_{t=n_{max}-n}^{n_{max}-1}[(e_{2n_{max}-n-t} a_n - f_{2n_{max}-n-t} b_n)\cos(tx) - (e_{2n_{max}-n-t} b_n + f_{2n_{max}-n-t} a_n)\sin(tx)] \end{aligned} \tag{12}$$

Thus, the effect of the nth mode of the first multiplier on the tth mode of the product is

$$\begin{aligned}
\left[Y_k Y_u\right]_{t,n} = &\frac{1}{2}[(e_{n-t}a_n + f_{n-t}b_n)\cos(tx) + (e_{n-t}b_n - f_{n-t}a_n)\sin(tx)] + \\
&\frac{1}{2}[(e_{t-n}a_n - f_{t-n}b_n)\cos(tx) + (e_{t-n}b_n + f_{t-n}a_n)\sin(tx)] + \\
&\frac{1}{2}[(e_{t+n}a_n + f_{t+n}b_n)\cos(tx) - (e_{t+n}b_n - f_{t+n}a_n)\sin(tx)] + \\
&\frac{1}{2}[(e_{2n_{max}-n-t}a_n - f_{2n_{max}-n-t}b_n)\cos(tx) - (e_{2n_{max}-n-t}b_n + f_{2n_{max}-n-t}a_n)\sin(tx)]
\end{aligned} \tag{13}$$

where Fourier components e_m and f_m with m out of the range $[0,n_{max}]$ are equal to zero.
This reduces to

$$\begin{aligned}
\left[Y_k Y_u\right]_{t,n} = &\left[\left(\frac{e_{n-t} + e_{t-n} + e_{t+n} + e_{2n_{max}-n-t}}{2}\right)a_n + \left(\frac{f_{n-t} - f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)b_n\right]\cos(tx) \\
+ &\left[\left(\frac{-f_{n-t} + f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)a_n + \left(\frac{e_{n-t} + e_{t-n} - e_{t+n} - e_{2n_{max}-t-n}}{2}\right)b_n\right]\sin(tx)
\end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned}
\left[Y_k \frac{dY_u}{dx}\right]_{t,n} = &\left[n\left(\frac{f_{n-t} - f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)a_n - n\left(\frac{e_{n-t} + e_{t-n} + e_{t+n} + e_{2n_{max}-n-t}}{2}\right)b_n\right]\cos(tx) \\
+ &\left[n\left(\frac{e_{n-t} + e_{t-n} - e_{t+n} - e_{2n_{max}-t-n}}{2}\right)a_n - n\left(\frac{-f_{n-t} + f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)b_n\right]\sin(tx)
\end{aligned} \tag{15}$$

$$\begin{aligned}
\left[Y_k \frac{d^2Y_u}{dx^2}\right]_{t,n} = &\left[-n^2\left(\frac{e_{n-t} + e_{t-n} + e_{t+n} + e_{2n_{max}-n-t}}{2}\right)a_n - n^2\left(\frac{f_{n-t} - f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)b_n\right]\cos(tx) \\
+ &\left[-n^2\left(\frac{-f_{n-t} + f_{t-n} + f_{t+n} - f_{2n_{max}-n-t}}{2}\right)a_n - n^2\left(\frac{e_{n-t} + e_{t-n} - e_{t+n} - e_{2n_{max}-t-n}}{2}\right)b_n\right]\sin(tx)
\end{aligned} \tag{16}$$

For example, consider the 2nd order linear ordinary differential equation with a forcing function D(x)

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = D(x) \tag{17}$$

We wish to solve for the Fourier components that would be generated by an FFT for the solution y based on the Fourier components generated by an FFT for A(x), B(x), C(x), and the forcing function, D(x) where

$$A(x) = \sum_{n=0}^{n_{max}}\left[Ac_n\cos(nx) + As_n\sin(nx)\right] \quad \text{(known)} \tag{18}$$

$$B(x) = \sum_{n=0}^{n_{max}} \left[Bc_n \cos(nx) + Bs_n \sin(nx) \right] \quad \text{(known)} \tag{19}$$

$$C(x) = \sum_{n=0}^{n_{max}} \left[Cc_n \cos(nx) + Cs_n \sin(nx) \right] \quad \text{(known)} \tag{20}$$

$$D(x) = \sum_{n=0}^{n_{max}} \left[Dc_n \cos(nx) + Ds_n \sin(nx) \right] \quad \text{(known)} \tag{21}$$

$$y(x) = \sum_{n=0}^{n_{max}} \left[yc_n \cos(nx) + ys_n \sin(nx) \right] \quad \text{(unknown)} \tag{22}$$

Substituting these five equations in the example equation (17) leads to the fact that the tth cosine mode of the LHS must equal the tth cosine mode of the RHS in the governing equation (17). Hence,

$$\sum_{n=0}^{n_{max}} \left\{ \begin{array}{l} \begin{bmatrix} Cc_{n-t} + Cc_{t-n} + Cc_{t+n} + Cc_{2n_{max}-n-t} \\ + n\left(Bs_{n-t} - Bs_{t-n} + Bs_{t+n} - Bs_{2n_{max}-n-t}\right) \\ - n^2\left(Ac_{n-t} + Ac_{t-n} + Ac_{t+n} + Ac_{2n_{max}-n-t}\right) \end{bmatrix} yc_n \\ + \begin{bmatrix} Cs_{n-t} - Cs_{t-n} + Cs_{t+n} - Cs_{2n_{max}-n-t} \\ - n\left(Bc_{n-t} + Bc_{t-n} + Bc_{t+n} + Bc_{2n_{max}-n-t}\right) \\ - n^2\left(As_{n-t} - As_{t-n} + As_{t+n} - As_{2n_{max}-n-t}\right) \end{bmatrix} ys_n \end{array} \right\} = 2Dc_t \tag{23}$$

Similarly, the tth sine mode of the LHS must equal the tth sine mode of the RHS in equation (17). Hence,

$$\sum_{n=0}^{n_{max}} \left\{ \begin{array}{l} \begin{bmatrix} - Cs_{n-t} + Cs_{t-n} + Cs_{t+n} - Cs_{2n_{max}-n-t} \\ + n\left(Bc_{n-t} + Bc_{t-n} - Bc_{t+n} - Bc_{2n_{max}-n-t}\right) \\ - n^2\left(- As_{n-t} + As_{t-n} + As_{t+n} - As_{2n_{max}-n-t}\right) \end{bmatrix} yc_n \\ + \begin{bmatrix} Cc_{n-t} + Cc_{t-n} - Cc_{t+n} - Cc_{2n_{max}-n-t} \\ - n\left(- Bs_{n-t} + Bs_{t-n} + Bs_{t+n} - Bs_{2n_{max}-n-t}\right) \\ - n^2\left(Ac_{n-t} + Ac_{t-n} - Ac_{t+n} - Ac_{2n_{max}-n-t}\right) \end{bmatrix} ys_n \end{array} \right\} = 2Ds_t \tag{24}$$

Thus, $2^*n_{max}+1$ linear equations can be solved for $2^*n_{max}+1$ unknown variables yc_0, yc_n, and ys_n that make up the analytical solution for y. Notice that no explicit boundary conditions are included in the system. Implicit boundary conditions arise from all basis functions being periodic. Thus, the final solution y will be periodic. In many cases, no additional boundary conditions need be applied to the system

3. Example of a Need for Boundary Conditions

However, consider the equation

$$y'' = 0 \tag{25}$$

solved using this method. We can get the analytic solution as

$$\begin{aligned} & y' = \text{const1} \\ & y = (\text{const1})x + \text{const2} \end{aligned} \tag{26}$$

As the basis fuctions guarantee a periodic solution y,

$$\begin{aligned} & y(0) = \text{const2} \\ & y(L) = (\text{const1})L + \text{const2} \\ & y(0) = y(L) \\ & \text{thus} \\ & \text{const1} = 0 \end{aligned} \tag{27}$$

However, for const2 to be determined uniquely, another boundary condition must be explicitly specified (perhaps y(0)). This equation deficiency will manifest itself in the set of equations as either a 0=0 equation or a set of linearly dependant equations. In either case, the system will not have sufficient rank, and will not be invertible. Therefore, care must be taken when examining the resulting system of equations (23) and (24) to determine when auxilliary boundary condition equations are needed.

One final note: this procedure can be likewise extended to higher order linear equations. However, the process results in increasingly stiff matrices of equations, due to the presence of powers of n in derivative terms. Thus, while n-squared appears in the above 2nd order equation, n-cubed would occur in a 3rd order equation, and so forth.

4. Summary

An analytical formulation was derived for integration of inhomogeneous ordinary differential equations with non-constant coefficients by utilizing a Fourier series method. Such equations can be used to model a distributed mass-damper-spring system oscillating and dampening while driven by an arbitrary distributed forcing function. This model is used for a robust inverse design of shapes of objects. The presented analytical formulation offer an enhanced convergence rate of the iterative determination of shapes when compared to the current model that uses constant mass-damper-spring coefficients.

5. References

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