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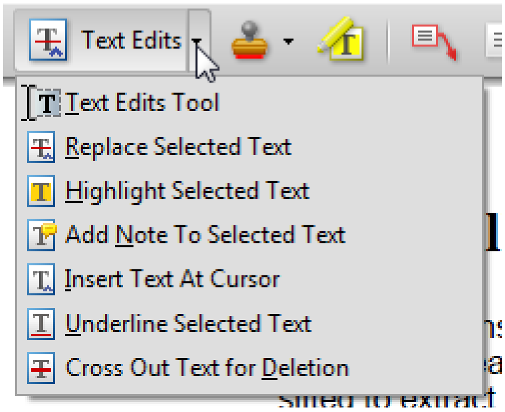

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On the inverse Noether's theorem in nonlinear micropolar continua

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In this article, the modified Noether's theorem is established in general form. Then, the inverse theorem is used in nonlinear micropolar continua in order to derive one-parameter family of transformations under which the corresponding functional is invariant. Next, the conservation laws are written. They include, as a special case, the conservation laws of micropolar elastostatics, the balance laws of elastodynamics and elastostatics. We do not analyse any of these special cases, because they may be obtained very easily.

Keywords: inverse Noether's theorem; nonlinear micropolar continua; family of transformations; conservation laws

AMS Subject Classifications: 74B20; 74A35; 74J25; 74D10

1. Introduction

In a paper by Knowles and Sternberg [1], it was shown that the conservation law, known as J-integral, that is,

$$J(G) = \mathbf{e} \cdot \int_G \left(W\mathbf{n} - \nabla \mathbf{x}^T \mathbf{T} \right) \mathrm{d}S, \tag{1}$$

(where G denotes any smooth non-self intersecting closed surface 'path of integration surrounding the crack', $\mathbf{n}$ is the unit outward normal vector on G, $\mathbf{e}$ is unit vector in the direction of the crack propagation, $\mathbf{T}$ is stress tensor, W is the strain-energy density at the point $\mathbf{x}$; generally by '$\cdot$', ∇ and T we shall denote the inner product, gradient and transpose, respectively, of corresponding quantities), follows from an application of Noether's theorem [1] on invariant variational principles to the principle of minimum potential energy in elastostatics. Roughly speaking, Noether's theorem states that if a given set of differential equations can be identified as the Euler-Lagrange equations corresponding to a variational principle which remains invariant under an n-parameter

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group of infinitesimal transformations, then there exists an associated set of n conservation laws satisfied by all solutions of the original differential equations. Moreover, this procedure yields two additional conservation laws stated in their Theorem 4.1.

Let D be a domain in E and let $[\mathbf{u}, \gamma, \sigma]$ be a finite elastic state on D, corresponding to the elastic potential Γ. Let W be the strain-energy density associated with Γ. Then, for every surface S, with the outward unit normal vector $\mathbf{n}$, that is the boundary of a regular subregion on D,

$$\int_S \left(Wn_i - s_j u_{j,i}\right) \mathrm{d}a = 0 \tag{2}$$

where $\mathbf{s}$ is Piola traction vector on S defined by $s_i = \sigma_{ij} n_j$ on S. If $[\mathbf{u}, \gamma, \sigma]$ is isotropic, then also

$$\int_S e_{ijk} \left(W x_k n_j + s_j u_k - s_p u_{p,j} x_k\right) \mathrm{d}a = 0. \tag{3}$$

Here, $\mathbf{u}$, γ and σ represent displacement field, its associated infinitesimal strain and stress field, respectively; e_{ijk} is the permutation symbol.

Knowles and Sternberg [1] also stated:

The completeness issue associated with the two conservation laws supplied by Theorem 4.1 *appears to be more complicated then the analogous question in the linearized theory*, which is answered by (their) Theorem 3.2.

Noether's theorem on variational principles invariant under a group of infinitesimal transformations was used also by Fletcher [2] to obtain a class of conservation laws associated with linear elastodynamics. These laws represent dynamical generalizations of path-independent integrals in elastodynamics. It is shown that the conservation laws obtained are the only ones obtainable by Noether's theorem from invariance under a reasonably general group of infinitesimal transformations.

In the mechanics of micropolar continua, this procedure was first used in [3]. In both papers, [2] and [3], the completeness of their results was established under certain conditions. It was done by the use of inverse Noether's theorem by which they were able to find a group of infinitesimal transformations. The procedures to find this group in those two papers are completely different. From the mathematical point of view, it is a real challenge and a more important part of Noether's theorem. In Fletcher's case it reads:

Suppose an elastic material under consideration is isotropic, and let the Lagrangian be given as

$$\mathcal{L}[\mathbf{w}] = \int_0^\tau \int_D L(\nabla \mathbf{w}, \dot{\mathbf{w}}) \mathrm{d}x \, \mathrm{d}t \tag{4}$$

where D is a bounded regular region in E_3, and where

$$L = \frac{1}{2} c_{ijkl} w_{i,j} w_{k,l} - \frac{1}{2} \varrho \dot{w}_i \dot{w}_i, \tag{5}$$

c_{ijkl} is elasticity tensor and ϱ is mass density. Then, $\mathcal{L}[\mathbf{w}]$ is infinitesimally invariant at $\mathbf{w}$ *under transformations of the form*

$$\xi^* = \Phi(\xi, \mathbf{w}(\xi); \varepsilon), \quad \mathbf{w}^* = \Psi(\xi, \mathbf{w}(\xi); \varepsilon), \quad \xi = (x_1, x_2, x_3, t) \tag{6}$$

for every w satisfying the Euler-Lagrange equations of motion for every D if and only if Φ and Ψ satisfy

$$\Phi(\xi, \mathbf{w}(\xi); \varepsilon) = \xi + \varepsilon\varphi(\xi) + \mathbf{o}(\varepsilon), \quad \Psi(\xi, \mathbf{w}(\xi); \varepsilon) = \mathbf{w} + \varepsilon\psi(\xi, \mathbf{w}(\xi)) + \mathbf{o}(\varepsilon), \tag{7}$$

as $\varepsilon \to 0$, where the components of φ and ψ are given by

$$\begin{aligned} \varphi_i &= \nu x_i + \varepsilon_{ijk} b_j x_k, \quad \varphi_4 = \nu t + c, \\ \psi_i &= -\nu w_i + \varepsilon_{ijk} b_j w_k + \varepsilon_{ijk} a_j x_k + d_i, \end{aligned} \tag{8}$$

and ν, c, b_i, a_i and d_i are arbitrary constants.

Note that dot over a quantity denotes time derivative.

We remark that the 'infinitesimal part' of the transformation $\xi \to \xi^*$, $\mathbf{w} \to \mathbf{w}^*$ is determined by the above transformations. This is all that is required for the corresponding conservation laws. In this way, we are able to see that six conservation laws are also valid in finite elasticity. Moreover, these conservation laws contain the principle of equivalence between conservation and invariance stated by Toupin [4] as:

Linear momentum, angular momentum, and energy are conserved in perfectly elastic medium if and only if the action density L is invariant under the group of Euclidean displacements.

Toupin did not use inverse Noether's theorem to derive the group of Euclidean displacements, but postulated them. Therefore, two important problems are left for the investigation:

(i) How to extend the use of Noether's theorem, and more important by the inverse Noether's theorem to a very general class of non-linear micropolar continuum, and
(ii) How to show that the completeness issue associated with the conservation laws we derived making use of inverse Noether's theorem, is not more complicated than the analogous question in the linear theory.

These topics will be the main purpose of this article.

This article is structured as follows: In Section 2, the notation to be used in the remainder and mathematical preliminaries of this article are introduced. In Section 3, the version of Noether's theorem appropriate for present purposes is stated. Section 4 contains a brief review of nonlinear micropolar continuum. Section 5 contains the principal results of inverse Noether's theorem that we discussed. The proof of the theorem is original. This theorem provides us with the generators of group of infinitesimal transformations. Then, in Section 6, the completeness of conservation laws is established. The conclusion is followed by the appendices. We point out the importance of Appendix C.

2. Mathematical preliminaries

Let ξ_α, $\alpha = 1, 2, \ldots, n$ be rectangular Cartesian coordinates in n-dimensional Euclidean space E_n, and let R be a bounded, closed regular region in E_n. With $\boldsymbol{\xi}(\xi_\alpha)$, we denote a point in $R \subset E_n$. Let $\boldsymbol{\Psi} = \left(\underset{\sigma}{\boldsymbol{\Psi}}\right), (\sigma = 1, 2, \ldots, k)$, be a set of tensor fields, defined and differentiable on R. Given the point $\xi(\xi_\alpha) \subset R$ and $\boldsymbol{\Psi} \subset C^2$, define a one-parameter η family of transformations $(\xi, \boldsymbol{\Psi}(\xi)) \Rightarrow (\bar{\xi}, \bar{\boldsymbol{\Psi}}(\bar{\xi}))$ as

$$\bar{\xi} = \bar{\xi}(\xi, \boldsymbol{\Psi}(\xi), \eta), \quad \bar{\boldsymbol{\Psi}} = \bar{\boldsymbol{\Psi}}(\xi, \boldsymbol{\Psi}(\xi), \eta) \tag{9}$$

For $\eta = 0$ these transformations are required to reduce to the identity

$$\bar{\xi}\big|_{\eta=0} = \xi, \quad \bar{\boldsymbol{\Psi}}\big|_{\eta=0} = \boldsymbol{\Psi}. \tag{10}$$

Hence, the infinitesimal transformations corresponding to (9) are given by

$$\begin{aligned} \bar{\xi} &= \xi + \boldsymbol{\varphi}\eta + \mathbf{O}(\eta^2), \quad \boldsymbol{\varphi} \stackrel{\text{def}}{=} \left(\frac{\partial \bar{\xi}}{\partial \eta}\right)_{\eta=0}, \\ \bar{\boldsymbol{\Psi}} &= \boldsymbol{\Psi} + \boldsymbol{\Phi}\eta + \mathbf{O}(\eta^2), \quad \boldsymbol{\Phi} \stackrel{\text{def}}{=} \left(\frac{\partial \bar{\boldsymbol{\Psi}}}{\partial \eta}\right)_{\eta=0} \end{aligned} \tag{11}$$

We shall distinguish the partial derivative $\partial_\xi \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial \xi_\alpha}\right) = (\partial_\alpha)$ from the total derivative $\nabla \stackrel{\text{def}}{=} (_{,\alpha})$ with respect to ξ. Also, by $\mathbf{D} = \nabla\bar{\xi} = \left\| \frac{\partial \bar{\xi}_\alpha}{\partial \xi_\beta} \right\|$ and $D = \det \mathbf{D}$ we denote the Jacobian of transformation '$\xi \Rightarrow \bar{\xi}$' and its determinant, respectively. From $(11)_1$, we obtain

$$\mathbf{D} = \mathbf{I} + (\nabla\boldsymbol{\varphi})\eta + \mathbf{O}(\eta^2). \tag{12}$$

Then

$$D_{\eta=0} = 1, \tag{13}$$

and, since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta} D &= \mathrm{Tr}\left[(\partial_{\nabla\bar{\xi}} D)\frac{\mathrm{d}}{\mathrm{d}\eta}(\nabla\bar{\xi})\right] = D\,\mathrm{Tr}\left[(\nabla\bar{\xi})\frac{\mathrm{d}}{\mathrm{d}\eta}(\nabla\bar{\xi})\right] \\ &= D\,\mathrm{Tr}\left[\mathbf{D}^{-1}\nabla\left(\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}\eta}\right)\right], \end{aligned} \tag{14}$$

where $\mathbf{D}^{-1}$ is inverse matrix of matrix $\mathbf{D}$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}\eta} D\right)_{\eta=0} = \nabla \cdot \boldsymbol{\varphi}. \tag{15}$$

The following sets

$$\mathbf{Y} = (\boldsymbol{\Psi}, \nabla\xi)(\xi), \tag{16}$$

$$\partial_{\mathbf{Y}} = (\partial_{\boldsymbol{\Psi}}, \partial_{\nabla\xi})(\xi) \tag{17}$$

will be used in the sequel.

3. Restricted version of Noether's theorem

Suppose that a real function $L(\xi, \mathbf{Y})$ is defined and differentiable for all values of its arguments. Now, let Λ be the functional defined by

$$\Lambda(\boldsymbol{\Psi}) = \int_R L(\xi, \mathbf{Y})\,\mathrm{d}\xi. \tag{18}$$

Definition

(a) The functional Λ is said to be invariant at $\boldsymbol{\Psi}$ under one-parameter family of transformations (9) if

$$\int_R L(\bar{\xi}, \bar{\mathbf{Y}})\,\mathrm{d}\bar{\xi} = \int_R L(\xi, \mathbf{Y})\,\mathrm{d}\xi \tag{19}$$

for all sufficiently small values of η.

(b) If, for a given value $\boldsymbol{\Psi}$,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\int_R L(\bar{\xi}, \bar{\mathbf{Y}})\,\mathrm{d}\bar{\xi}\right)_{\eta=0} = 0, \tag{20}$$

then Λ is said to be infinitesimally invariant at $\boldsymbol{\Psi}$.

Evidently, if Λ is invariant at $\boldsymbol{\Psi}$, then Λ is infinitesimally invariant at $\boldsymbol{\Psi}$. On the basis of the introduced notations, it is possible to state a restricted version of Noether's theorem, as follows:

Theorem 3.1 *If $\boldsymbol{\Psi}$ satisfies the Euler-Lagrange equations*

$$\nabla \cdot \partial_{\nabla\boldsymbol{\Psi}} L - \partial_{\boldsymbol{\Psi}} L = 0 \tag{21}$$

then Λ is an infinitesimally invariant at $\boldsymbol{\Psi}$ under transformations (9) *iff $\boldsymbol{\Psi}$ defined in R, too, satisfies*

$$\nabla \cdot \left\{L\boldsymbol{\varphi} + (\partial_{\nabla\boldsymbol{\Psi}} L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}]\right\} = 0. \tag{22}$$

If ∂R is the boundary of R, and if its unit outward normal vector is $\mathbf{N}$, then (22) *can be written in the form*

$$\int_{\partial R} \left\{L\boldsymbol{\varphi} + (\partial_{\nabla\boldsymbol{\Psi}} L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}]\right\} \cdot \mathbf{N}\,\mathrm{d}S = 0. \tag{23}$$

Proof First, by $\nabla \cdot \partial_{\nabla\boldsymbol{\Psi}}$ we denote the composition with respect to ∇, that is

$$\nabla \cdot \partial_{\nabla\boldsymbol{\Psi}} = \left(\frac{\partial(\cdot)}{\partial\boldsymbol{\Psi}_{,\alpha}}\right)_{,\alpha} \quad \text{obviously} \quad (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi} = \boldsymbol{\Psi}_{,\alpha}\varphi_\alpha. \tag{24}$$

We are going to prove the part 'if' of the theorem. Then (20) holds. It may be written in an equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\int_R L(\bar{\xi}, \bar{\mathbf{Y}})\,D\mathrm{d}\xi\right)_{\eta=0} = 0, \tag{25}$$

from which we obtain, in view of (13) and (15),

$$\left[D\frac{\mathrm{d}}{\mathrm{d}\eta}L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})+L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})\frac{\mathrm{d}}{\mathrm{d}\eta}D\right]_{\eta=0}=0 \tag{26}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\eta}L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})_{\eta=0}+L(\boldsymbol{\xi},\mathbf{Y})\nabla\cdot\boldsymbol{\varphi}=0. \tag{27}$$

However,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})_{\eta=0}&=\left[\partial_{\bar{\boldsymbol{\xi}}}L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})\cdot\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\boldsymbol{\xi}}\right]_{\eta=0}+\left[\partial_{\bar{\mathbf{Y}}}L(\bar{\boldsymbol{\xi}},\bar{\mathbf{Y}})\cdot\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\mathbf{Y}}\right]_{\eta=0}\\&=\partial_{\boldsymbol{\xi}}L(\boldsymbol{\xi},\mathbf{Y})\cdot\boldsymbol{\varphi}+\partial_{\mathbf{Y}}L(\boldsymbol{\xi},\mathbf{Y})\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\mathbf{Y}}\right)_{\eta=0}.\end{aligned} \tag{28}$$

Now,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\mathbf{Y}}\right)_{\eta=0}=\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\boldsymbol{\Psi}},\ \frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\nabla}\bar{\boldsymbol{\Psi}}\right)_{\eta=0}=[\boldsymbol{\Psi},\ \nabla\boldsymbol{\Psi}-(\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\Psi}], \tag{29}$$

since

$$\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\boldsymbol{\Psi}}\right)_{\eta=0}=\left\{\frac{\mathrm{d}}{\mathrm{d}\eta}[(\nabla\bar{\boldsymbol{\Psi}})\mathbf{D}^{-1}]\right\}_{\eta=0}=\nabla\boldsymbol{\Psi}-(\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\Psi} \tag{30}$$

because $\mathbf{D}^{-1}_{\eta=0}=\mathbf{I}$ and $\left(\frac{\mathrm{d}}{\mathrm{d}\eta}D^{-1}\right)_{\eta=0}=-\nabla\cdot\boldsymbol{\varphi}$ in view of (13). It is instructive to write $(\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\Psi}$ in more explicit form (which is also useful to clarify the notation), i.e. in the from $(\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\Psi}=(\boldsymbol{\Psi}_{,\beta}\varphi_{\beta,\alpha})$. Also,

$$\begin{aligned}\nabla\cdot(L\boldsymbol{\varphi})&=(\nabla L)\cdot\boldsymbol{\varphi}+L\nabla\cdot\boldsymbol{\varphi}=\partial_{\boldsymbol{\xi}}L\cdot\boldsymbol{\varphi}+\boldsymbol{\varphi}\cdot(\nabla\mathbf{Y})\partial_{\mathbf{Y}}L+L\nabla\cdot\boldsymbol{\varphi}\\&=\partial_{\boldsymbol{\xi}}L\cdot\boldsymbol{\varphi}+(\boldsymbol{\varphi}\cdot\nabla)\mathbf{Y}\cdot\partial_{\mathbf{Y}}L+L\nabla\cdot\boldsymbol{\varphi},\end{aligned} \tag{31}$$

so that

$$\begin{aligned}&\nabla\cdot(L\boldsymbol{\varphi})+\left[\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\mathbf{Y}}\right)_{\eta=0}-(\boldsymbol{\varphi}\cdot\nabla)\mathbf{Y}\right]\cdot\partial_{\mathbf{Y}}L\\&\quad=\nabla\cdot(L\boldsymbol{\varphi})+\partial_{\boldsymbol{\Psi}}L\cdot[\boldsymbol{\Phi}-(\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}]+\partial_{\nabla\boldsymbol{\Psi}}L\cdot\nabla[\boldsymbol{\Phi}-(\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}],\end{aligned} \tag{32}$$

or

$$\left\{L\boldsymbol{\varphi}+(\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi}-(\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}]\right\}+(\partial_{\boldsymbol{\Psi}}L-\nabla\cdot\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi}-(\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}]=0. \tag{33}$$

From this expression we obtain (22) taking into account (21). ■

Remark 1 The part 'only if' of the theorem is known as the inverse Noether's theorem and depends on the form of conservation laws. This will be considered in the case of micropolar continua.

Remark 2 It may happen that there exist some fields, say $\boldsymbol{\omega} = \binom{\omega}{\tau}$, $\tau = 1, 2, \ldots, \pi$, under which

$$\Lambda(\boldsymbol{\Psi}, \boldsymbol{\omega}) = \int_R L(\xi, \mathbf{Y})\,\mathrm{d}\xi, \quad \mathbf{Y} = (\boldsymbol{\Psi}, \boldsymbol{\omega}, \nabla\boldsymbol{\Psi}, \nabla\boldsymbol{\omega})(\xi), \tag{34}$$

is infinitesimally invariant, but otherwise does not satisfy Euler-Lagrange equations. Then, we shall speak of one-parameter family of transformations

$$T_\eta\colon\ (\xi, \boldsymbol{\Psi}, \boldsymbol{\omega}) \Rightarrow (\bar{\xi}, \bar{\boldsymbol{\Psi}}, \bar{\boldsymbol{\omega}}),$$

and, in addition to (11) (depending now also on $\boldsymbol{\omega}$), we shall have

$$\bar{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varsigma\eta + \mathbf{O}(\eta^2), \quad \varsigma = \left(\frac{\mathrm{d}\bar{\boldsymbol{\omega}}}{\mathrm{d}\eta}\right)_{\eta=0}. \tag{35}$$

If this is the case (33) has to be modified

$$\begin{aligned}\nabla\cdot\big\{L\boldsymbol{\varphi} + (\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + (\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}]\big\}\\ + (\partial_{\boldsymbol{\Psi}}L - \nabla\cdot\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + (\partial_{\boldsymbol{\omega}}L - \nabla\cdot\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}] = 0.\end{aligned} \tag{36}$$

Then, we may state the modified Noether's theorem, as follows:

THEOREM 3.2 *If* $\boldsymbol{\Psi}$ *satisfies the Euler-Lagrange equations* (21) *then* Λ, *defined by* (34), *is infinitesimally invariant at* $(\boldsymbol{\Psi}, \boldsymbol{\omega})$ *under extended transformations* (11) *and* (35) *if* $(\boldsymbol{\Psi}, \boldsymbol{\omega})$ *defined in R satisfies also*

$$\begin{aligned}\nabla\cdot\big\{L\boldsymbol{\varphi} + (\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + (\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}]\big\}\\ + (\partial_{\boldsymbol{\omega}}L - \nabla\cdot\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}] = 0.\end{aligned} \tag{37}$$

If ∂R *is the boundary of R, and if its unit outward normal is* $\mathbf{n}$, *then* (37) *can be written as follows*

$$\begin{aligned}\int_{\partial R}\big\{L\boldsymbol{\varphi} + (\partial_{\nabla\boldsymbol{\Psi}}L)[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + (\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}]\big\}\mathbf{n}\,\mathrm{d}S\\ + \int_R(\partial_{\boldsymbol{\omega}}L - \nabla\cdot\partial_{\nabla\boldsymbol{\omega}}L)[\varsigma - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\omega}]\,\mathrm{d}V = 0.\end{aligned} \tag{38}$$

Proof The proof of the theorem is straightforward. For further references, we shall write (36) in the following form

$$\begin{aligned}\partial_\xi L\cdot\boldsymbol{\varphi} + L\nabla\cdot\boldsymbol{\varphi} + \partial_{\boldsymbol{\Psi}}L\cdot\boldsymbol{\Phi} + \partial_{\boldsymbol{\omega}}L\cdot\varsigma\\ + \partial_{\nabla\boldsymbol{\Psi}}L\cdot[\nabla\boldsymbol{\Phi} - (\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\Psi}] + \partial_{\nabla\boldsymbol{\omega}}L\cdot[\nabla\varsigma - (\nabla\boldsymbol{\varphi})\nabla\boldsymbol{\omega}] = 0,\end{aligned} \tag{39}$$

or

$$\begin{aligned}\frac{\partial L}{\partial\xi_\alpha}\varphi_{,\alpha} + L\varphi_{\alpha,\alpha} + \partial_{\boldsymbol{\Psi}}L\cdot\boldsymbol{\Phi}\partial_{\boldsymbol{\omega}}L\cdot\varsigma\\ + \partial_{\boldsymbol{\Psi}_{,\alpha}}L\cdot(\boldsymbol{\Phi}_{,\alpha} - \boldsymbol{\Psi}_{,\beta}\,\varphi_{\beta,\alpha}) + \partial_{\boldsymbol{\omega}_{,\alpha}}L\cdot(\varsigma_{,\alpha} - \boldsymbol{\omega}_{,\beta}\,\varphi_{\beta,\alpha}) = 0.\end{aligned} \tag{40}$$

This expression represents the starting point of the inverse Noether's theorem.

As an important example in considering the part of inverse Noether's theorem we shall deal with the micropolar continua. The advantage of micropolar continua and main difference between it and classical continua is given below.

4. Micropolar elastic continuum

Classical continuum theories assume that stress at a material point is dependent on, for example, temperature, strain, strain rate and strain history at the same point. Due to the lack of information on neighbouring material points, these models are called local. However, when the microscopic and macroscopic length scales are comparable, the assumption is questionable as the material behavior at a point is influenced by the deformation of neighbouring points. To incorporate the scale of the microstructure of a heterogeneous material within the continuum framework, a number of phenomenological remedies have been proposed that involve the relaxation of the local action hypothesis of classical continuum mechanics.

In the micropolar or 'Cosserat' continuum models, independent rotational degrees of freedom are introduced at the point of the continuum in addition to the displacement field. In this manner, curvatures and couple stresses account for the effect of neighbouring material points. The general theory of simple micropolar-elastic solids has been formulated by Eringen and Suhubi [5]. Here, we recall certain results from the theory of finitely deformed homogeneous and isotropic elastic solids in the absence of body forces and body couples, we are going to use.

The space of events is E_3. For the sake of generality, here we use the spatial x^k and the material coordinates X^K, respectively, whereby all quantities are expressed as functions of X^K and time t.

The motion of micropolar continuum is described by

$$x^k = x^k\left(X^K, t\right), \tag{41}$$

$$\chi^k{}_K = \chi^k{}_K\left(X^K, t\right), \tag{42}$$

where orthogonal tensor $\chi^k{}_K$ – is microrotation tensor.

The material forms of the balance laws of momentum, moment of momentum and energy are, respectively,

$$T^{Kk}{}_{;K} = \varrho_0 \ddot{x}^k, \tag{43}$$

$$\varepsilon^{klm}\left(M^{KL}{}_{k;L} - S^K{}_k\right)\chi_{mK} = \varrho_0 \dot{\sigma}^l, \tag{44}$$

$$-\varrho_0\dot{\varepsilon} + T^K{}_k \dot{x}^k{}_{;K} + S^K{}_k \dot{\chi}^k{}_K + M^{KL}{}_k \dot{\chi}^k{}_{K;L} = 0, \tag{45}$$

where ϱ_0 – mass density in the reference configuration, T^{Kk} – stress tensor, $M^{KL}{}_k$ – couple stress tensor, ε – internal energy density, σ^k – spin density, ε_{klm} – permutation symbol, and

$$S^K{}_k = T^{Ll} x_{k;L} \chi_l{}^K + \frac{1}{2}\varepsilon_{klm} M^{Ll} \chi^{mK}{}_{;L}, \tag{46}$$

$$M^{Ll} = \varepsilon^{klm} M^{KL}{}_k \chi_{mK}, \quad M^{KL}{}_k = \frac{1}{2}\varepsilon_{klm} M^{Ll} \chi^{mK}, \tag{47}$$

$$\sigma^k = j^{kl}\nu_l, \quad J_{KL} = j_{kl}\chi^k{}_K \chi^l{}_L, \quad j^{kl} = J^{KL}\chi^k{}_K \chi^l{}_L, \tag{48}$$

$$\nu_k = -\frac{1}{2}\varepsilon_{klm}\nu^{lm}, \quad \nu_{kl} = \dot{\chi}_{kK}\chi_l{}^K, \quad \varepsilon_{klm}\chi^k{}_K \chi^l{}_L \chi^m{}_M = \varepsilon_{KLM}. \tag{49}$$

The constitutive equations of micropolar elastic material are [5]

$$T^K{}_k = \varrho_0 \frac{\partial \varepsilon}{\partial x^k{}_{;K}}, \tag{50}$$

$$S^K{}_k = \varrho_0 \frac{\partial \varepsilon}{\partial \chi^k{}_K}, \tag{51}$$

$$M^{KL}{}_k = \varrho_0 \frac{\partial \varepsilon}{\partial \chi^k{}_{K;L}}. \tag{52}$$

5. The inverse Noether's theorem

It is easy to see that (43) can be written as

$$\left(\frac{\partial \Sigma}{\partial x^k{}_{;K}}\right)_{;K} - \varrho_0 \ddot{x}^k = 0, \tag{53}$$

where $\Sigma = \varrho_0 \varepsilon$.

Further on, for the sake of simplicity and clarity, we shall use Cartesian coordinates. Then, we may write e_{klm} instead of ε_{klm}. Also, if we state

$$\xi^\alpha = \begin{cases} X^K & \text{for } \alpha = K = 1, 2, 3, \\ t & \text{for } \alpha = 4, \end{cases} \tag{54}$$

we may write concisely

$$(x_{k,\alpha}) = (x_{k,K}, \dot{x}_k), \tag{55}$$

$$(\chi_{k\alpha}) = (\chi_{kK}, \dot{\chi}_{kK}). \tag{56}$$

Let

$$L = \Sigma - \frac{1}{2}\varrho_0 \dot{x}_k \dot{x}_k - \frac{1}{2}\varrho_0 j_{kl}\nu_k \nu_l. \tag{57}$$

So defined function L dependents on the set of variables $(x_{k,\alpha}, \chi_{kK}, \chi_{kK,\alpha})$, i.e. $L = L(x_{k,\alpha}, \chi_{kK}, \chi_{kK,\alpha})$. Then, $\frac{\partial L}{\partial \xi_\alpha} = 0$. With this definition of function L, (53) can be written more elegantly and simply as Euler-Lagrange equations

$$\left(\frac{\partial L}{\partial x_{k;\alpha}}\right)_{;\alpha} = 0. \tag{58}$$

Generally, this is not true for the balance laws of moment of momentum (44) and energy (45), i.e. they can not be written in the form of Euler-Lagrange equations. This suggests that we should try to apply modified Noether's theorem in order to find the family of transformations under which functional (34) defined by function (57) should be invariant. If so, we have to deal with (40). Then, the following identifications are necessary

$$\boldsymbol{\Psi} = \mathbf{x}(x_k), \quad \nabla\boldsymbol{\Psi} = \nabla\mathbf{x}(x_{k;\alpha}), \quad \boldsymbol{\omega} = \boldsymbol{\chi}(\chi_{kK}), \quad \nabla\boldsymbol{\omega} = \nabla\boldsymbol{\chi}(\chi_{kK,\alpha}) \tag{59}$$

in view of which (40) becomes

$$\begin{aligned} L\varphi_{\alpha,\alpha} + \frac{\partial L}{\partial x_{k;\alpha}}\left(\Phi_{k,\alpha} - x_{k,\beta}\,\varphi_{\beta,\alpha}\right) + \frac{\partial L}{\partial \chi_{kK}}\zeta_{kK} \\ + \frac{\partial L}{\partial \chi_{kK,\alpha}}\left(\zeta_{kK,\alpha} - \chi_{kK,\beta}\,\varphi_{\beta,\alpha}\right) = 0. \end{aligned} \tag{60}$$

From this equation, under certain conditions we are going to impose, we shall try to find the generators of infinitesimal transformations φ_α, Φ_k, ζ_{kK}. In fact, these conditions are imposed by the principle of material objectivity, which must be satisfied by all material, and the principle of material invariance [6]. Because of this, the functional form of the internal energy ε or function Σ cannot be arbitrary. More precisely, if we impose the condition that the form of Σ must satisfy material objectivity, but otherwise remains arbitrary, we must have

$$\varepsilon_{klm}\left(\frac{\partial\Sigma}{\partial x_{k,K}}x_{l,K} + \frac{\partial\Sigma}{\partial\chi_{kK}}\chi_{lK} + \frac{\partial\Sigma}{\partial\chi_{kK,M}}\chi_{lK,M} + \frac{\partial\Sigma}{\partial\chi_{kM,K}}\chi_{kM,L}\right) = 0. \tag{61}$$

If the material is initially isotropic, according to the principle of material invariance, ε or function Σ, in addition to (61), must satisfy the conditions

$$\varepsilon_{KLN}\left(\frac{\partial\Sigma}{\partial x_{k,K}}x_{k,L} + \frac{\partial\Sigma}{\partial\chi_{kK}}\chi_{kK} + \frac{\partial\Sigma}{\partial\chi_{kK,M}}\chi_{kL,M} + \frac{\partial\Sigma}{\partial\chi_{kM,K}}\chi_{kM,L}\right) = 0. \tag{62}$$

These two set of conditions are of great importance in deriving the appropriate family of transformations under which the functional Λ is invariant. We proceed to find their generators explicitly.

We state explicitly, for simple homogeneous micropolar-elastic solids in the absence of body forces and body couples, the following

Theorem 5.1 *The functional Λ, given by* (34) *for L defined by* (57), *is infinitesimally invariant at* $(\mathbf{x}, \boldsymbol{\chi})(\xi)$ *if*

$$\begin{aligned} &\begin{cases} \varphi_K = \varepsilon_{KLM}X_L A_M + C_K, \\ \varphi_4 = A, \end{cases} \\ &\Phi_k = \varepsilon_{klm}x_l\,a_m + c_k, \\ &\zeta_{kK} = \varepsilon_{klm}\chi_{lK}\,a_m, \end{aligned} \tag{63}$$

where A, A_K, C_K, a_k and c_k are arbitrary constants.

Proof The proof is based on Lagrange multipliers method. We start with (60), i.e. with

$$L\,\varphi_{\alpha,\alpha} + \frac{\partial L}{\partial x_{k,K}}\left(\Phi_{k,K} - x_{k,\beta}\,\varphi_{\beta;K}\right) + \frac{\partial L}{\partial \dot{x}_k}\left(\dot{\Phi}_k - x_{k,\beta}\,\dot{\varphi}_\beta\right)$$
$$+ \frac{\partial L}{\partial \chi_{kK}}\zeta_{kK,L} + \frac{\partial L}{\partial \chi_{kK}}\left(\zeta_{kK,L} - \chi_{kK,\beta}\,\varphi_{,L}\right) + \frac{\partial L}{\partial \dot{\chi}_{kK}}\left(\dot{\zeta}_{kK} - \chi_{kK,\beta}\,\dot{\varphi}_L\right) = 0. \qquad (64)$$

From (57) we calculate

$$\frac{\partial L}{\partial x_{k,K}} = \frac{\partial \Sigma}{\partial x_{k,K}}, \quad \frac{\partial L}{\partial \dot{x}_k} = -\varrho_0 \dot{x}_k, \quad \frac{\partial L}{\partial \chi_{kK,L}} = \frac{\partial \Sigma}{\partial \chi_{kK,L}}. \qquad (65)$$

However,

$$\frac{\partial L}{\partial \chi_{kK}} = \frac{\partial \Sigma}{\partial \chi_{kK}} - \frac{1}{2}\varrho_0 \frac{\partial j_{pq}\nu_p\nu_q}{\partial \chi_{kK}} \quad \text{and} \quad \frac{\partial L}{\partial \dot{\chi}_{kK}} = -\frac{1}{2}\varrho_0 \frac{\partial j_{pq}\nu_p\nu_q}{\partial \dot{\chi}_{kK}}. \qquad (66)$$

After some lengthy calculations (Appendix A) we find that

$$\frac{\partial j_{pq}\nu_p\nu_q}{\partial \chi_{kK}} = \sigma_p\left(\chi_{pK}\nu_k + \chi_{kK}\nu_p\right) = \sigma_p\left(2\chi_{pK}\nu_k - \varepsilon_{prk}\,\dot{\chi}_{rK}\right) \qquad (67)$$

and

$$\frac{\partial j_{pq}\nu_p\nu_q}{\partial \dot{\chi}_{kK}} = \varepsilon_{prk}\sigma_p\,\chi_{rK}. \qquad (68)$$

Then

$$\frac{\partial L}{\partial \chi_{kK}} = \frac{\partial \Sigma}{\partial \chi_{kK}} - \frac{1}{2}\varrho_0\sigma_p\left(\chi_{pK}\nu_k + \chi_{kK}\nu_p\right) \quad \text{and} \quad \frac{\partial L}{\partial \dot{\chi}_{kK}} = -\frac{1}{2}\varrho_0\varepsilon_{prk}\sigma_p\,\chi_{rK}. \qquad (69)$$

Hence, in the view of (61) and (62), (64) can be written in the form

$$L\,\varphi_{\alpha,\alpha} + \frac{\partial \Sigma}{\partial x_{k,K}}\left(\Phi_{k,K} - x_{k,\beta}\,\varphi_{,K}\right) - \varrho_0\dot{x}_k\left(\dot{\Phi}_k - x_{k,\beta}\,\dot{\varphi}_\beta\right)$$
$$+ \left[\frac{\partial \Sigma}{\partial \chi_{kK}} - \frac{1}{2}\varepsilon_0\sigma_p\left(2\chi_{pK}\nu_k - \varepsilon_{prk}\,\dot{\chi}_{rK}\right)\right]\zeta_{kK}$$
$$+ \frac{\partial \Sigma}{\partial \chi_{kK,L}}\left(\zeta_{kK,L} - \chi_{kK,\beta}\,\varphi_{\beta,L}\right) - \frac{1}{2}\varrho_0\varepsilon_{prk}\sigma_p\,\chi_{rK}\left(\dot{\zeta}_{kK} - \chi_{kK,\beta}\,\dot{\varphi}_\beta\right)$$
$$- l_m\varepsilon_{klm}\left(\frac{\partial \Sigma}{\partial x_{k,K}}x_{l,K} + \frac{\partial \Sigma}{\partial \chi_{kK}}\chi_{lK} + \frac{\partial \Sigma}{\partial \chi_{kK,L}}\chi_{lK,L}\right)$$
$$- \Lambda_N\,\varepsilon_{KLN}\left(\frac{\partial \Sigma}{\partial x_{k,K}}x_{l,L} + \frac{\partial \Sigma}{\partial \chi_{kK}}\chi_{kL} + \frac{\partial \Sigma}{\partial \chi_{kK,M}}\chi_{kL,M} + \frac{\partial \Sigma}{\partial \chi_{kM,K}}\chi_{kM,L}\right) = 0 \qquad (70)$$

where λ_m and Λ_M are Lagrange multipliers and generally may depend on the set of variable $(x_{k,K}, \chi_{kK}, \chi_{kK,L})$.

Now we come to the main point of our work. We postulate that (70) must be satisfied for arbitrary

$$\Sigma,\ \frac{\partial \Sigma}{\partial x_{k,K}},\ \frac{\partial \Sigma}{\partial \chi_{kK}},\ \frac{\partial \Sigma}{\partial \chi_{kK,L}}. \qquad (71)$$

As a consequence of this postulate, it follows that the coefficients of these quantities in (70) must be zero. Thus,

$$\Sigma: \ \varphi_{\alpha,\alpha} = 0, \tag{72}$$

$$\frac{\partial \Sigma}{\partial x_{k,K}}: \ \Phi_{k,K} - x_{k,\beta}\,\varphi_{\beta,K} - \lambda_m \varepsilon_{klm}\, x_{l,K} - \Lambda_N\, \varepsilon_{KLN}\, x_{k,L} = 0, \tag{73}$$

$$\frac{\partial \Sigma}{\partial \chi_{kK}}: \ \zeta_{kK} - l_m\, \varepsilon_{klm}\, \chi_{lK} - \Lambda_N\, \varepsilon_{KLN}\, \chi_{kL} = 0, \tag{74}$$

$$\begin{aligned} \frac{\partial \Sigma}{\partial \chi_{kK,L}}: \ & \zeta_{kK,L} - \chi_{kK,\beta}\,\varphi_{\beta,L} - l_m\, \varepsilon_{klm}\, \chi_{lK,L} \\ & - \Lambda_N\, \varepsilon_{KMN}\, \chi_{kM,L} - \Lambda_N\, \varepsilon_{LMN}\, \chi_{kK,M} = 0, \end{aligned} \tag{75}$$

and (70) reduces to

$$\begin{aligned} \dot{x}_k\big(\dot{\Phi}_k - x_{k,\beta}\,\dot{\varphi}_\beta\big) + \frac{1}{2}\sigma_p\big(2\chi_{pK}\nu_k - \varepsilon_{prk}\,\dot{\chi}_{rK}\big)\chi_{kK} \\ + \frac{1}{2}\varepsilon_{prk}\sigma_p\,\chi_{rK}\big(\dot{\zeta}_{kK} - \chi_{kK,\beta}\,\dot{\varphi}_\beta\big) = 0. \end{aligned} \tag{76}$$

Next, we write explicitly expressions for $\varphi_{\alpha,\beta}$, $\Phi_{l,\alpha}$, $\zeta_{kK,\alpha}$, i.e.

$$\varphi_{\alpha,\beta} = \frac{\partial \varphi_\alpha}{\partial \xi_\beta} + \frac{\partial \varphi_\alpha}{\partial x_k} x_{k,\beta} + \frac{\partial \varphi_\alpha}{\partial \chi_{kK}} \chi_{kK,\beta}, \tag{77}$$

$$\Phi_{l,\beta} = \frac{\partial \Phi_l}{\partial \xi_\beta} + \frac{\partial \Phi_l}{\partial x_k} x_{k,\beta} + \frac{\partial \Phi_l}{\partial \chi_{kK}} \chi_{kK,\beta} \tag{78}$$

$$\zeta_{kK,\beta} = \frac{\partial \zeta_{kK}}{\partial \xi_\beta} + \frac{\partial \zeta_{kK}}{\partial x_k} x_{k,\beta} + \frac{\partial \zeta_{kK}}{\partial \chi_{lL}} \chi_{lL,\beta}. \tag{79}$$

We require that (72)–(76) hold for any value of $x_{k,\beta}$, $\chi_{kK,\beta}$.

(1) Then it results from (72) that

$$\frac{\partial \varphi_\alpha}{\partial \xi_\alpha} = 0, \quad \frac{\partial \varphi_\alpha}{\partial \chi_{kK}} = 0, \quad \text{i.e.} \quad \varphi_\alpha = \varphi_\alpha(\xi_\beta) \tag{80}$$

so that

$$\frac{\partial \varphi_\alpha}{\partial \xi_\alpha} = 0. \tag{81}$$

(2) From (74) we have trivially

$$\zeta_{kK} = l_m \varepsilon_{klm}\, \chi_{lK} + \Lambda_N \varepsilon_{KLN} \chi_{kL}. \tag{82}$$

(3) Equation (76) gives two equations

$$\dot{x}_k\big(\dot{\Phi}_k - x_{k,\beta}\dot{\varphi}_\beta\big) = 0, \tag{83}$$

$$\sigma_p\big(2\chi_{pK}\nu_k - \varepsilon_{prk}\dot{\chi}_{rK}\big)\zeta_{kK} + \varepsilon_{prk}\sigma_p\chi_{rK}\big(\dot{\zeta}_{kK} - \chi_{kK,\beta}\,\dot{\varphi}_\beta\big) = 0. \tag{84}$$

We write (83) as

$$\dot{x}_k\left(\frac{\partial \Phi_k}{\partial t} + \frac{\partial \Phi_k}{\partial x_l}\dot{x}_l + \frac{\partial \Phi_k}{\partial \chi_{lK}}\dot{\chi}_{lK} - x_{k,K}\,\dot{\varphi}_K - \dot{x}_k\dot{\varphi}_4\right) = 0. \tag{85}$$

From this, we obtain that

$$\dot{\varphi}_K = 0 \Rightarrow \varphi_K = \varphi_K(X_L), \tag{86}$$

in the view of (82). Also

$$\frac{\partial \Phi_k}{\partial t} = 0, \quad \frac{\partial \Phi_k}{\partial \chi_{lK}} = 0, \tag{87}$$

and

$$\frac{\partial \Phi_{(k}}{\partial x_{l)}} = \frac{\partial \varphi_4}{\partial t}\delta_{kl}, \tag{88}$$

where ( ) designates the symmetrization of the indices k and l.

We shall analyse (84) later.

(4) We proceed with (73), i.e. with

$$\begin{aligned}\frac{\partial \Phi_k}{\partial X_K} + \frac{\partial \Phi_k}{\partial x_l}x_{l,K} + \frac{\partial \Phi_k}{\partial \chi_{lL}}\chi_{lL,K} - x_{k,L}\varphi_{L,K} - \dot{x}_k\frac{\partial \varphi_4}{\partial X_K}\\ - l_m\,\varepsilon_{klm}x_{l,K} - \Lambda_N\varepsilon_{KLN}x_{k,L} = 0,\end{aligned} \tag{89}$$

which reduces to

$$\frac{\partial \Phi_k}{\partial X_K} + \frac{\partial \Phi_k}{\partial x_l}x_{l,K} - x_{k,L}\varphi_{L,K} - \dot{x}_k\frac{\partial \varphi_4}{\partial X_K} - l_m\varepsilon_{klm}x_{l,K} - \Lambda_N\varepsilon_{KLN}x_{k,L} = 0. \tag{90}$$

Because of (87), from the above equations, it follows that:

$$\frac{\partial \Phi_k}{\partial X_K} = 0 \Rightarrow \Phi_k = \Phi_k(x_l), \tag{91}$$

$$\frac{\partial \varphi_4}{\partial X_K} = 0 \Rightarrow \varphi_4 = \varphi_4(t), \tag{92}$$

and

$$\left(\frac{\partial \Phi_k}{\partial x_l} - l_m\varepsilon_{klm}\right)x_{l,K} - \big(\varphi_{L,K} + \Lambda_N\varepsilon_{KLN}\big)x_{k,L} = 0. \tag{93}$$

But, this is equivalent to

$$\frac{\partial \Phi_k}{\partial x_l} - l_m\,\varepsilon_{klm} - \big(\varphi_{L,K} + \Lambda_N\varepsilon_{KLN}\big)x_{k,L}X_{K,l} = 0, \tag{94}$$

from which we obtain

$$\frac{\partial \Phi_{(k}}{\partial x_{l)}} - l_m\,\varepsilon_{klm} - \big(\varphi_{L,K} + \Lambda_N\,\varepsilon_{KLN}\big) X_{K,(l} x_{k),L} = 0. \tag{95}$$

Substituting (88) into it, we get

$$\frac{\partial \varphi_4}{\partial t}\,\delta_{kl} - \big(\varphi_{L,K} + \Lambda_N \varepsilon_{KLN}\big) X_{K,(l} x_{k),L} = 0, \tag{96}$$

from which we obtain

$$\frac{\partial \varphi_4}{\partial t} = 0 \Rightarrow \varphi_4 = A. \tag{97}$$

But, then (88) reads

$$\frac{\partial \Phi_{(k}}{\partial x_{l)}} = 0, \tag{98}$$

from which we have

$$\Phi_k = \varepsilon_{klm} x_l a_m + c_k, \tag{99}$$

where a_m and c_k are arbitrary constants. From (99) and (93) we get

$$(a_m - l_m)\varepsilon_{klm} x_{l,K} - \big(\varphi_{L,K} + \Lambda_N \varepsilon_{KLN}\big) x_{k,L} = 0, \tag{100}$$

and from this

$$\varphi_{(K,L)} = 0 \Rightarrow \varphi_K = \varepsilon_{KLM} X_L A_M + C_K, \tag{101}$$

where A_K and C_K are arbitrary constants. Then, in view of this, the above equations can be written as

$$(a_m - l_m)\varepsilon_{klm} x_{l,K} = (\Lambda_N - A_N)\varepsilon_{KLN} x_{k,L} = 0, \tag{102}$$

and obviously this is satisfied if

$$l_m = a_m \quad \text{and} \quad \Lambda_N = A_N. \tag{103}$$

(5) Now, we pass our investigation to (75), i.e. to

$$\begin{aligned} &\frac{\partial \zeta_{kK}}{\partial X_L} + \frac{\partial \zeta_{kK}}{\partial x_l} x_{l,K} + \frac{\partial \zeta_{kK}}{\partial \chi_{lM}} \chi_{lM,L} - \chi_{kK,M}\varphi_{M,L} - \dot{\chi}_{kK}\varphi_{4,L} \\ &\quad - l_m \varepsilon_{klm}\chi_{lK,L} - \Lambda_N \varepsilon_{KMN}\chi_{kM,L} - \Lambda_N \varepsilon_{LMN}\chi_{kK,M} = 0, \end{aligned} \tag{104}$$

which is reduced to

$$\begin{aligned} &\frac{\partial \zeta_{kK}}{\partial \chi_{lM}} \chi_{lM,L} - \chi_{kK,M}\varphi_{M,L} - l_m \varepsilon_{klm}\chi_{lK,L} \\ &\quad - \Lambda_N \varepsilon_{KMN}\chi_{kM,L} - \Lambda_N \varepsilon_{LMN}\chi_{kK,M} = 0, \end{aligned} \tag{105}$$

since

$$\frac{\partial \zeta_{kK}}{\partial X_L} = 0, \qquad \frac{\partial \zeta_{kK}}{\partial x_l} = 0, \tag{106}$$

and $\varphi_{4,L} = 0$ due to (97). This residual equality is identity in view of (82) and (101).

Therefore, the only condition which is left for the investigation is (84). It reduces to

$$\sigma_p\big[\big(2\chi_{pK}v_k - \varepsilon_{prk}\dot{\chi}_{rK}\big)\zeta_{kK} + \varepsilon_{prk}\chi_{rK}\dot{\zeta}_{kK}\big] = 0 \tag{107}$$

by means of (101). After very lengthy calculations (Appendix B), making use of the previous results, we are able to write it as

$$A_M\sigma_p\dot{\chi}_{pM} = 0, \tag{108}$$

so that

$$A_M\sigma_p = 0 \tag{109}$$

must hold. Thus, $A_M = 0$ and

$$\zeta_{kK} = l_m\varepsilon_{klm}\chi_{lK}. \tag{110}$$

Remark 1 The condition $A_M\sigma_p\dot{\chi}_{pM} = 0$ is obviously satisfied when $\dot{\chi}_{pM} = 0 \Rightarrow \mu_p = 0 \Rightarrow \sigma_p = 0$. Then, we have to modify our theorem, i.e. in this case

$$\begin{gathered}
\begin{cases} \varphi_K = \varepsilon_{KLM}X_LA_M + C_K, \\ \varphi_4 = A, \end{cases} \\
\Phi_k = \varepsilon_{klm}x_la_m + c_k \\
\zeta_{kK} = a_m\varepsilon_{klm}\chi_{lK} + A_M\varepsilon_{KLM}\chi_{kL},
\end{gathered} \tag{111}$$

since there is no restriction on the value of A_M. Moreover, L satisfies Euler-Lagrange equations which are nothing else but (44) (Appendix C).

6. Conservation laws

Now, we proceed to write the integral form of the conservation law (38) of micropolar continuum in the absence of external body force density and external body couple density having in mind that $(\xi_\alpha) = (X_m, t)$ and L is defined by (57). The other quantities we have to use are given by (59). Further, it is convenient to write (63) in the form

$$\boldsymbol{\varphi} = \begin{cases} \mathbf{f} = \mathbf{X}\wedge\mathbf{A} + \mathbf{C}, \\ \varphi_4 = A, \end{cases} \tag{112}$$

$$\boldsymbol{\Phi} = \mathbf{x}\wedge\mathbf{a} + \mathbf{c} \tag{113}$$

$$\boldsymbol{\zeta} = \boldsymbol{\chi}\wedge\mathbf{a}. \tag{114}$$

We shall use (36) for further reference. Then,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_V\big\{L\varphi_4 + \partial_{\dot{\mathbf{x}}}L[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + \partial_{\dot{\boldsymbol{\chi}}}[\boldsymbol{\zeta} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}]\big\}\mathrm{d}V \\
&\quad + \int_S\big\{L\mathbf{f} + \partial_{\mathrm{Grad}\mathbf{x}}L[\boldsymbol{\Phi} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\Psi}] + \partial_{\mathrm{Grad}\boldsymbol{\chi}}L[\boldsymbol{\zeta} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}]\big\}\cdot\mathbf{N}\mathrm{d}S \\
&\quad + \int_V\big\{\big(\partial_{\boldsymbol{\chi}}L - \nabla\cdot\partial_{\mathrm{Grad}\boldsymbol{\chi}}L\big)[\boldsymbol{\zeta} - (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\zeta}]\big\}\mathrm{d}V = 0
\end{aligned} \tag{115}$$

where Grad denotes a gradient with respect to X_K. In component form it reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left[L\varphi_4+\frac{\partial}{\partial\dot{x}_k}L\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)+\frac{\partial}{\partial\dot{\chi}_{kK}}L\big(\zeta_{kK}-\chi_{kK,\alpha}\varphi_\alpha\big)\right]\mathrm{d}V\\&\quad+\int_S\left[Lf_K+\frac{\partial}{\partial x_{k,K}}L\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)+\frac{\partial}{\partial\chi_{kl,K}}L\big(\zeta_{kL}-\chi_{kL,\alpha}\varphi_\alpha\big)\right]N_K\mathrm{d}S\\&\quad+\int_V\left[\left(\frac{\partial L}{\partial\chi_{kK}}-\left(\frac{\partial L}{\partial\chi_{kK,\beta}}\right)_{,\beta}\right)\big(\zeta_{kK}-\chi_{kL,\alpha}\varphi_\alpha\big)\right]\mathrm{d}V=0.\end{aligned}\tag{116}$$

In the view of the expressions for stress and couple stress tensors, this may be written as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left[L\varphi_4-\varrho_0\dot{x}_k\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)-\frac{1}{2}\varrho_0\varepsilon_{prk}\sigma_p\chi_{rK}\big(\zeta_{kK}-\chi_{kK,\alpha}\varphi_\alpha\big)\right]\mathrm{d}V\\&\quad+\int_S\big[Lf_K+T_{Kk}\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)+M_{LKk}\big(\zeta_{kL}-\chi_{kL,\alpha\varphi_\alpha}\big)\big]N_K\mathrm{d}S\\&\quad-\int_V\big[\varrho_0\sigma_p\nu_p\chi_{kK}\big(\zeta_{kK}-\chi_{kK,\alpha}\varphi_\alpha\big)\big]\mathrm{d}V=0.\end{aligned}\tag{117}$$

since

$$\partial_{\chi_{kK}}L-\big(\partial_{\chi_{kK,\alpha}}L\big)_{,\alpha}=-\varrho_0\sigma_p\nu_p\chi_{kK}\tag{118}$$

(Appendix C). This may be further simplified to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left[L\varphi_4-\varrho_0\dot{x}_k\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)-\frac{1}{2}\varrho_0\varepsilon_{prk}\sigma_p\chi_{rK}\big(\zeta_{kK}-\chi_{kK,\alpha}\varphi_\alpha\big)\right]\mathrm{d}V\\&\quad+\int_S\big[Lf_K+T_{Kk}\big(\Phi_k-x_{k,\alpha}\varphi_\alpha\big)+M_{LKk}\big(\zeta_{kL}-\chi_{kL,\alpha\varphi_\alpha}\big)\big]N_K\mathrm{d}S=0,\end{aligned}\tag{119}$$

taking into account that $\chi_{kK}\chi_{kK,\alpha}=0$ and $\chi_{kK}\zeta_{kK}=0$ (Appendix B).

By taking all of the arbitrary constants **a**, **c**, **A**, A, **C** in (112)–(114) to be zero except one, in turn, we obtain the corresponding conservation law. There are five transformations under which the corresponding functional Λ is infinitesimally invariant.

6.1. *Spatial invariance*

(I) Under translation

$$\boldsymbol{\Phi}=\mathbf{c},\quad\boldsymbol{\varphi}=\mathbf{0},\quad\boldsymbol{\zeta}=\mathbf{0}\tag{120}$$

gives balance of momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V\varrho_0x_k\mathrm{d}V=\int_S T_{Kk}N_K\mathrm{d}S.\tag{121}$$

(II) Under rotation

$$\boldsymbol{\Phi} = \mathbf{x} \wedge \mathbf{a}, \quad \boldsymbol{\zeta} = \boldsymbol{\chi} \wedge \mathbf{a}, \quad \boldsymbol{\varphi} = \mathbf{0} \tag{122}$$

gives balance of moment of momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0(\varepsilon_{klm}x_l\dot{x}_m + \sigma_k)\mathrm{d}V = \int_S (\varepsilon_{klm}x_l T_{Km} + M_{Kk})N_K \mathrm{d}S. \tag{123}$$

6.2. *Shift of time*

Conditions

$$\varphi_4 = A, \quad \mathbf{f} = \mathbf{0}, \quad \boldsymbol{\Phi} = \mathbf{0}, \quad \boldsymbol{\zeta} = \mathbf{0} \tag{124}$$

give the balance of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(L + \varrho_0\dot{x}_k\dot{x}_k + \frac{1}{2}\varrho_0\varepsilon_{prk}\sigma_p\chi_{rK}\dot{\chi}_{kK}\right)\mathrm{d}V = \int_S (T_{Kk}\dot{x}_k + M_{LKk}\dot{\chi}_{kL})N_K\mathrm{d}S. \tag{125}$$

However, by virtue of Appendix A and definition of M_{Kk}, this expression reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0\left(\varepsilon + \frac{1}{2}\dot{x}_k\dot{x}_k + \frac{1}{2}\sigma_k\nu_k\right)\mathrm{d}V = \int_S (T_{Kk}\dot{x}_k + M_{Kk}\nu_k)N_K\mathrm{d}S. \tag{126}$$

6.3. *Material invariance*

(I) Under translation (homogeneous material)

$$\mathbf{f} = \mathbf{C}, \quad \varphi_4 = 0, \quad \boldsymbol{\Phi} = \mathbf{0}, \quad \boldsymbol{\zeta} = \mathbf{0} \tag{127}$$

gives conservation of linear material momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0\left(\dot{x}_k x_{k,L} + \frac{1}{2}\varepsilon_{prk}\sigma_p\chi_{rK}\chi_{kK,L}\right)\mathrm{d}V + \int_S \left(L\delta_{KL} - T_{Kk}x_{k,L} - M_{MKk}\chi_{kM,L}\right)N_K\mathrm{d}S = 0. \tag{128}$$

(II) Under rotation (isotropic material)

$$f_K = \varepsilon_{KLM}X_L A_M, \quad \varphi_4 = 0, \quad \boldsymbol{\Phi} = \mathbf{0}, \quad \zeta_{kK} = 0 \tag{129}$$

we have the following integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f_M \varrho_0 \left(\dot{x}_k x_{k,M} + \frac{1}{2}\varepsilon_{prk}\sigma_p \chi_{rK}\chi_{kK,M} \right) \mathrm{d}V + \int_S f_M \left(L\delta_{KM} - T_{Kk}x_{k,M} - M_{LKk}\chi_{kL,M} \right) N_K \mathrm{d}S = 0, \tag{130}$$

from which we obtain conservation of angular material momentum.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0 \varepsilon_{MPQ} X_P \left(\dot{x}_k x_{k,M} + \frac{1}{2}\varepsilon_{prk}\sigma_p \chi_{rK}\chi_{kK,M} \right) \mathrm{d}V + \int_S \varepsilon_{MPQ} X_P \left(L\delta_{KM} - T_{Kk}x_{k,M} - M_{LKk}\chi_{kL,M} \right) N_K \mathrm{d}S = 0 \tag{131}$$

(III) In a special case when

$$\zeta_{kK} = \varepsilon_{KLM}\chi_{kL}A_M, \quad f_K = \varepsilon_{KLM}X_L A_M, \quad \varphi_4 = 0, \quad \mathbf{\Phi} = \mathbf{0}, \tag{132}$$

the function L, given by (57), must be modified, i.e.

$$L = \Sigma - \frac{1}{2}\varrho_0 \dot{x}_k \dot{x}_k. \tag{133}$$

Then, the following integral is obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f_M \varrho_0 \dot{x}_k x_{k,M} \mathrm{d}V + \int_S \left[f_M \left(L\delta_{KM} - T_{Kk}x_{k,M} - M_{LKk}\chi_{kL,M} \right) + M_{MKk}\zeta_{kM} \right] N_K \mathrm{d}S = 0, \tag{134}$$

or, finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0 \varepsilon_{MPQ} X_P \dot{x}_k x_{k,M} \mathrm{d}V + \int_S \left[\varepsilon_{MPQ} X_P \left(L\delta_{KM} - T_{Kk}x_{k,M} - M_{LKk}\chi_{kL,M} \right) + M_{Kp}\chi_{pQ} \right] N_K \mathrm{d}S = 0. \tag{135}$$

7. Conclusion

A rigorous framework for the Noether's theorem has been presented. This framework is sufficiently general to be applied for different classes of materials from the view point of continuum mechanics. We emphasize the importance of the inverse Noether's theorem in order to derive family of transformations under which the functional Λ is invariant. In fact, once we know the family of transformations under which functional Λ is invariant, it is easy to obtain conservation laws. Therefore, the more important and more challenging part of Noether's theorem is the inverse Noether's theorem.

It is fully demonstrated in the theory of micropolar continuum.

Among six conservation laws so derived, last three of them constitute new material balance laws in a sense they depend on the family of transformations given in reference configuration. They include, as a special case, the conservation laws of micropolar elastostatics, and the balance laws of elastodynamics as well as elastostatics (see [2,7,8]). We do not analyse any of these special cases because they may be obtained very easily. These laws find the applications in fracture mechanics.

The application of the inverse Noether's theorem in micropolar continuum also clarifies the difference between necessity and sufficiency of Noether's theorem and the completeness issue associated with the conservation laws we derived.

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1,4 **References**

[1] J.K Knowles and E. Sternberg, *On a class of conservation laws in linearized and finite elasticity*, Arch. Rational Mech. Anal. 44 (1972), p. 187.
[2] D.C. Fletcher, *Conservation laws in linear elastodynamics*, Arch. Rational Mech. Anal. 60 (1976), p. 329.
[3] J. Jarić, *Conservation laws of J-integral type in micropolar elastostatics*, Int. J. Eng. Sci. 16 (1978), pp. 967–984.
[4] R.A. Toupin, *Theory of elasticity with couple-stress*, Arch. Rational Mech. Anal. 17 (1964), p. 113.
[5] C.A. Eringen and E.S. Suhubi, *Nonlinear theory of simple micro-elastic solids-I*, Int. J. Engng Sci. 2 (1954), pp. 189–203.
[6] C.A. Eringen, *Polar and Nonpolar Field Theories*, *Continuum Physics*, Vol. IV, Academic Press, New York, San Francisco, London, 1976.
[7] J.R. Rice, *A path independent integral and the approximate analysis of strain concentrations by notches and cracks*, J. Appl. Mech. 35(2) (l963), p. 379.
[8] E. Noether, *Invariante Variationsprobleme*, Grittinger Nachrichten 3 (1871), p. 351.
2,3 [9] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, in *Handbuch der Physik*, Vol. III/3, [redacted], ed., Springer Verlag, Berlin-Gettingen-Heidelberg, 1960, pp. [redacted].

Appendix A

First, from

$$\nu_p = -\frac{1}{2} e_{prk} \dot{\chi}_{rK} \chi_{kK} \tag{A1}$$

we have

$$\frac{\partial \nu_p}{\partial \chi_{kK}} = -\frac{1}{2} e_{prk} \dot{\chi}_{rK}. \tag{A2}$$

However,

$$\nu_{kl} = \dot{\chi}_{kK} \chi_{lK} = -e_{klm} \nu_m \tag{A3}$$

and hence

$$\dot{\chi}_{kK} = -e_{klm}v_m\chi_{lK}, \tag{A4}$$

so that

$$e_{prk}\dot{\chi}_{rK} = \chi_{pK}v_k - \chi_{kK}v_p. \tag{A5}$$

Thus,

$$\frac{\partial v_p}{\partial \chi_{kK}} = \frac{1}{2}\left(\chi_{pK}v_k - \chi_{kK}v_p\right). \tag{A6}$$

Further,

$$j_{pq} = J_{PQ}\chi_{pP}\chi_{qQ} \quad \text{and} \quad J_{PQ} = j_{pq}\chi_{pP}\chi_{qQ}, \tag{A7}$$

so that

$$\frac{\partial j_{pq}}{\partial \chi_{kK}}v_p v_q = J_{PQ}\frac{\partial \chi_{pP}\chi_{qQ}}{\partial \chi_{kK}}v_p v_q = 2\sigma_p\chi_{pK}v_k. \tag{A8}$$

Now, it is easy to see that

$$\begin{aligned}\frac{\partial j_{pq}v_p v_q}{\partial \chi_{kK}} &= \frac{\partial j_{pq}}{\partial \chi_{kK}}v_p v_q + 2\sigma_p\frac{\partial v_p}{\partial \chi_{kK}}\\ \sigma_p\left(2\chi_{pK}v_k - e_{prk}\dot{\chi}_{rK}\right) &= \sigma_p\left(v_p\chi_{kK} + v_k\chi_{pK}\right),\end{aligned} \tag{A9}$$

which is (67).

Next, from

$$\frac{\partial v_p}{\partial \dot{\chi}_{kK}} = -\frac{1}{2}e_{pkr}\chi_{rK} \tag{A10}$$

we obtain

$$\frac{\partial j_{pq}v_p v_q}{\partial \dot{\chi}_{kK}} = 2\sigma_p\frac{\partial v_p}{\partial \dot{\chi}_{kK}} = e_{prk}\sigma_p\chi_{rK}, \tag{A11}$$

which is (68).

Appendix B

We calculate in detail

$$\begin{aligned}\chi_{rK}\zeta_{kK} &= \chi_{rK}(a_m\varepsilon_{klm}\chi_{lK} + A_M\varepsilon_{KLM}\chi_{kL})\\ &= a_m\varepsilon_{klm}\chi_{lK}\chi_{rK} + A_M\varepsilon_{KLM}\chi_{rK}\chi_{kL}\\ &= a_m\varepsilon_{krm} + A_M e_{rkm}\chi_{mM} = e_{rkm}(A_M\chi_{mM} - a_m).\end{aligned} \tag{B1}$$

Obviously

$$\chi_{kK}\zeta_{kK} = 0 \tag{B2}$$

and

$$\dot{\overline{\chi_{rK}\zeta_{kK}}} = e_{rkm}A_M\dot{\chi}_{mM}. \tag{B3}$$

Also,

$$e_{prk}\dot{\overline{\chi_{rK}\zeta_{kK}}} = e_{prk}e_{rkm}A_M\dot{\chi}_{mM} = 2A_M\dot{\chi}_{pM}. \tag{B4}$$

Now,

$$\sigma_p\left[2\left(\chi_{pK}\nu_k - \varepsilon_{prk}\dot{\chi}_{rK}\right)\zeta_{kK} + \varepsilon_{prk}\chi_{rK}\dot{\zeta}_{kK}\right] = 0 \tag{B5}$$

may be written as

$$\sigma_p\left[2\left(\chi_{pK}\nu_k - \varepsilon_{prk}\dot{\chi}_{rK}\right)\zeta_{kK} + \varepsilon_{prk}\chi_{rK}\dot{\zeta}_{kK}\right] = 0, \tag{B6}$$

or

$$\sigma_p\left[2\left(\chi_{pK}\zeta_{kK}\right)\nu_p + 2A_M\dot{\chi}_{pM}\right] = 0, \tag{B7}$$

and finally

$$A_M\sigma_p\dot{\chi}_{pM} = 0, \tag{B8}$$

which is nothing but (108).

Appendix C

We write

$$\begin{aligned}
\partial_{\chi_{kK}}L - \left(\partial_{\chi_{kK,\alpha}}L\right)_{,\alpha} &= \partial_{\chi_{kK}}L - \left(\partial_{\chi_{kK,L}}L\right)_{,L} - \dot{\overline{\partial_{\chi_{kK}}L}} \\
&= \partial_{\chi_{kK}}\Sigma - \left(\partial_{\chi_{kK,L}}\Sigma\right)_{,L} - \frac{1}{2}\varrho_0\sigma_p\left(2\chi_{pK}\nu_p - e_{prk}\dot{\chi}_{rK}\right) + \frac{1}{2}\varrho_0 e_{prk}\dot{\overline{\sigma_p\chi_{rK}}} \\
&= \partial_{\chi_{kK}}\Sigma - \left(\partial_{\chi_{kK,L}}\Sigma\right)_{,L} + \frac{1}{2}\varrho_0 e_{prk}\dot{\sigma}_p\chi_{rK} - \varrho_0\sigma_p\left(\chi_{pK}\nu_p - e_{prk}\dot{\chi}_{rK}\right) \\
&= \partial_{\chi_{kK}}\Sigma - \left(\partial_{\chi_{kK,L}}\Sigma\right)_{,L} + \frac{1}{2}\varrho_0 e_{prk}\dot{\sigma}_p\chi_{rK} - \varrho_0\sigma_p\chi_{pK}\nu_p \\
&= S_{Kk} - M_{KLk,L} + \frac{1}{2}\varrho_0 e_{prk}\dot{\sigma}_p\chi_{rK} - \varrho_0\sigma_p\chi_{kK}\nu_p.
\end{aligned} \tag{C1}$$

However, as shown in [5]

$$e_{klm}\left(M_{KLk,L} - S_{Kk}\right)\chi_{mK} = \varrho_0\dot{\sigma}_l. \tag{C2}$$

Multiplying both sides of this expression by e_{plm}, we have

$$S_{Kk} - M_{KLk,L} + \frac{1}{2}\varrho_0 e_{prk}\dot{\sigma}_p\chi_{rK} = 0. \tag{C3}$$

Substituting this in the expression above, we finally obtain Lagrange equations of the second kind

$$\partial_{\chi_{kK}}L - \left(\partial_{\chi_{kK,\alpha}}L\right)_{,\alpha} = -\varrho_0\sigma_p\nu_p\chi_{kK}, \tag{C4}$$

which is (118).