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Stokes' Hypothesis and Entropy Variation Within a Compression Shock

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Abstract. Exact one-dimensional steady flow equations for viscous, heat conducting calorically perfect gases were derived in terms of a velocity potential. These non-linear differential equations were integrated numerically thus predicting detailed variation of thermodynamic and flow properties through the normal steady compression shock waves for an entire range of ratios of secondary and primary viscosity coefficients, Reynolds numbers, and upstream Mach numbers. The results conform that entropy reaches its sharp maximum inside the shock wave and that shock wave strengths and losses correspond to Rankine-Hugoniot jump conditions only when Stokes hypothesis (zero bulk viscosity) is enforced.

1 Introduction

The difference between the thermodynamic pressure, p, and the mechanical (or hydraulic pressure or average normal stress) pressure, $\tilde{p}$, can be expressed (to the first order approximation) as

$$p - \tilde{p} = \mu_{\mathrm{B}} \left(\nabla \cdot \vec{V} \right) \tag{1}$$

Here, p is the thermodynamic pressure (p = ρRT) and $\tilde{p}$ is the average normal total stress (hydraulic pressure) which can be expressed as

$$\tilde{p} = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = p - \left(\lambda + \frac{2}{3}\mu \right) \left(\nabla \cdot \vec{V} \right) \tag{2}$$

where the factor $\mu_B = \lambda + \frac{2}{3}\mu$ is often termed the coefficient of bulk viscosity. Thus, the mean total pressure in the deforming viscous fluid is not equal to the thermodynamic property called pressure. This distinction is rarely important, since divergence of velocity vector is usually very small in typical flow problems. In 1854, Stokes himself simply resolved the issue by an assumption known as Stokes hypothesis. It states that the bulk coefficient of viscosity is zero which gives the following value for the second coefficient of viscosity

$$\lambda = -\frac{2}{3}\mu \tag{3}$$

The Stokes hypothesis has been a controversial subject for more than a century. It has been recognized that the actual values of the second coefficient of viscosity differ often significantly from the values obtained from Eq. 3 [1]. It has also been recognized that the ratio of the second and first coefficients of viscosity, λ/μ, can have a profound effect on the strength of a compression shock waves and its thickness [2,3]. The objective of this paper is to show these effects by deriving appropriate differential equations and integrating them numerically. The results will expose the inner structure of normal steady shocks as they depend on values of λ/μ that differ from those suggested by Stokes hypothesis [4,5,6].

2 Analysis

Entropy generation form of the dimensional energy conservation equation for homocompositional fluids without heat radiation and no body forces and having constant thermal conductivity, can be written as

$$\bar{\rho}\bar{T}\left(\frac{\partial\ \bar{s}}{\partial\ \bar{t}}+\left(\bar{\vec{V}}\bullet\bar{\nabla}\right)\bar{s}\right)=\bar{\Phi}+\bar{k}\bar{\nabla}^2\bar{T} \tag{4}$$

where $\bar{\Phi}$ is the viscous dissipation function and s is the entropy per unit mass. If the flow is unidirectional, we can utilize a velocity potential function, ϕ, such that $\vec{V}=\nabla\phi$. Coefficients of second viscosity, molecular viscosity, and thermal conductivity, $\lambda,\mu,$ and k, respectively, will be treated as constants. Let the characteristic dimensional flow quantities be designated with the subscript * and the local dimensional quantities have an overbar. Non-dimensionalization can then be performed as follows.

$$\begin{gathered}\rho=\frac{\bar{\rho}}{\bar{\rho}_*},\quad T=\frac{\bar{T}}{\bar{T}_*},\quad a^2=\frac{\bar{a}^2}{\bar{a}_*^2},\quad \phi_S=\frac{\left|\bar{\vec{V}}\right|}{\bar{a}_*}=M_*,\quad x=\frac{\bar{x}}{\bar{L}},\quad \mathrm{s}=\frac{\bar{\mathrm{s}}}{\bar{\mathrm{R}}},\quad \mathrm{C_p}=\frac{\bar{C}_p}{\bar{C}_{p*}}\\ k=\frac{\bar{k}}{\bar{k}_*},\quad \mu=\frac{\bar{\mu}}{\bar{\mu}_*},\quad \lambda=\frac{\bar{\lambda}}{\bar{\mu}_*},\quad \mu_{\mathrm{L}}=(2+\frac{\bar{\lambda}}{\bar{\mu}})\frac{\bar{\mu}}{\bar{\mu}_*},\quad \mathrm{Re}=\frac{\bar{\rho}_*\bar{a}_*\bar{L}}{\bar{\mu}_*},\quad \mathrm{Pr}=\frac{\bar{\mathrm{C}}_{\mathrm{p}*}\bar{\mu}_*}{\bar{k}_*}\end{gathered} \tag{5}$$

In isoenergetic flows, the local temperature normalized with the critical temperature is

$$\frac{\bar{T}}{\bar{T}_*}=\left[\frac{\gamma+1}{2}-\frac{\gamma+1}{2}M_*^2\right] \tag{6}$$

The non-dimensionalized local speed then becomes the local characteristic Mach number

$$\phi_x = \left(\partial \bar{\phi} / \partial \bar{x}\right) / \bar{a}_* = M_* \tag{7}$$

Laplacian of the normalized local temperature in terms of the velocity potential derivatives is then

$$\nabla^2 T = \nabla^2 \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \left(\nabla\phi \bullet \nabla\phi \right) \right] \tag{8}$$

In one-dimensional case, this reduces to

$$\frac{\partial^2 T}{\partial x^2} = \left[-(\gamma - 1)\left({\phi_{xx}}^2 + \phi_x \phi_{xxx} \right) \right] \tag{9}$$

which, when substituted into the energy conservation equation, gives the entropy generation equation

$$\mathrm{Re}\, \rho T \phi_x \frac{ds}{dx} = \gamma\, \mu_L \left(\phi_{xx}\right)^2 - \frac{\gamma\, k\, \bar{k}_* (\gamma - 1)}{\gamma\, \bar{R} \bar{\mu}_*} \left[\phi_x \phi_{xxx} + \left(\phi_{xx}\right)^2 \right] \tag{10}$$

For non-isentropic flows, local non-dimensional pressure can be expressed in terms of the velocity potential and normalized entropy change.

$$\rho = \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2 \right]^{\frac{1}{\gamma - 1}} \mathrm{e}^{-\Delta \mathrm{s}} \tag{11}$$

Notice that

$$\frac{(\gamma - 1)}{\gamma \bar{R}} = \frac{1}{\bar{C}_p} \tag{12}$$

so that the entropy gradient across a shock in terms of the velocity potential can be obtained by integrating

$$\frac{ds}{dx} = \frac{\gamma}{\mathrm{Re}} \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \left(\phi_x\right)^2 \right]^{\frac{-\gamma}{\gamma - 1}} \mathrm{e}^{\Delta \mathrm{s}} \left[\left(\mu_L - \frac{k}{\mathrm{Pr}} \right) \frac{\left(\phi_{xx}\right)^2}{\phi_x} - \frac{k}{\mathrm{Pr}} \phi_{xxx} \right] \tag{13}$$

Since this equation has two unknowns (s and ϕ) it has to be solved together with the one-dimensional steady state version of the novel three-dimensional unsteady compressible viscous heat conducting flow Physically Dissipative full Potential (PDP) equation [4,5,6].

$$\rho\left[1-\left(\frac{\phi_x}{a}\right)^2\right]\phi_{\mathrm{xx}}=\frac{1}{\mathrm{Re}}\left[\frac{\gamma-1}{a^2}\left(\mu_L-\frac{k}{\mathrm{Pr}}\right)(\phi_{xx})^2-\frac{\phi_x}{a^2}\left(\mu_L+(\gamma-1)\frac{k}{\mathrm{Pr}}\right)\phi_{xxx}\right] \quad (14)$$

Equations (13) and (14) need to be integrated simultaneously to obtain variation of entropy and other thermodynamic and flow quantities through the shock and their jump conditions across the shock. For comparison, the exact values of the total entropy jump across a normal steady compression shock satisfying Rankine-Hugoniot conditions can be found from

$$\Delta s_{2-1}=\frac{\gamma}{\gamma-1}\ln\left[\frac{2}{(\gamma+1)M_1{}^2}+\frac{\gamma-1}{\gamma+1}\right]+\frac{1}{\gamma-1}\ln\left[\frac{2\gamma}{\gamma+1}M_1{}^2-\frac{\gamma-1}{\gamma+1}\right] \quad (15)$$

where M_1 is the Mach number at upstream infinity and Δs_{2-1} is the jump in entropy between downstream infinity and upstream infinity.

3 Numerical Results

Equation (14) was integrated numerically to predict variation of entropy, speed, density, pressure, and temperature inside normal steady compression shock waves. Nondimensional density was computed from Eq. (121) and local change of nondimensional entropy was computed from Eq. (13). Numerical integration of Eq. (14), which is a truly nonlinear ordinary differential equation of third order, was performed using a fourth-order Runge-Kutta scheme and several values of $(\phi_x)_{-\infty}$, $\overline{\lambda}/\overline{\mu}$, and Re.

Figure 1 depicts computed variation of Mach number through the normal steady shock wave for the ratio of secondary to shear viscosity λ/μ = -0.6666, three values of upstream Mach number (M1 = 1.2, 1.6 and 2.0), and three values of Reynolds number (thick line Re = 500,000, dashed line Re = 1,000,000, and thin line Re = 5,000,000). It is evident that the total change of the Mach number through the shock does not depend on Reynolds number. However, thickness of the shock significantly reduces with increase of the Reynolds number and especially with the increase of the shock strength.

Figure 2 depicts computed variation of nondimensionalized entropy through the normal steady shock wave for the ratio of secondary to shear viscosity that corresponds to Stokes hypothesis (λ/μ = -0.6666), three values of upstream Mach number (M1 = 1.2, 1.6 and 2.0), and three values of Reynolds number (thick line Re = 500,000, dashed line Re = 1,000,000, and thin line Re = 5,000,000). It is evident that the total change of entropy through the shock does not depend on Reynolds number. However, variation of entropy inside the shock exhibits a very sharp spike as analyzed analytically by Morduchow and Libby [3] and numerically by the author [4,5,6]. This phenomenon can be explained as follows. From Eq. (4) it is evident that entropy increases in the upstream portion of the shock wave because viscous dissipation and second derivative of temperature (head conduction term) are both positive there.

In the downstream portion of the shock the viscous dissipation remains positive, but sign of the second derivative of temperature becomes negative thus making strong negative contributions of heat conduction to the entropy change. Eventually, the entropy far downstream of the shock must be greater than the entropy far upstream of the shock of second law of thermodynamics would be violated. However, entropy inside the shock can locally decrease significantly as evidenced in Fig. 2.

Figure 3 demonstrates a potentially important effect of departing from Stokes hypothesis. Namely, Mach number far downstream of a normal steady compression shock can become much lower (that is, the shock could become much stronger than the Rankine-Hugoniot shock) if the ratio of viscosities λ / μ could be made smaller than –2/3. If this ratio could be made larger than –2/3, the shock would become slightly weaker. These trends become more pronounced with the increase of the upstream Mach number.

Figure 4 however shows that creating stronger shocks by somehow reducing the ratio of viscosities λ / μ would incur a significant increase of entropy until the value of approximately λ / μ = - 1.8 is reached. By lowering the value of λ / μ further towards λ / μ = - 2.0 the entropy jump across such shocks would rapidly decrease and eventually tend toward values similar to the Rankine-Hugoniot entropy jump conditions. It appears that shock waves much stronger than Rankine-Hugoniot jump conditions are possible while having practically the same losses (entropy jumps) if ratio of the viscosities λ / μ could be made as close to λ / μ = - 2.0 as possible. This corresponds to zero normal viscous stresses situation.

On the other hand, shocks that are only slightly weaker than Rankine-Hugoniot shocks should be possible o generate with significantly lower losses (entropy jumps) than Rankine-Hugoniot shocks the viscosity ratio λ / μ could be made large and positive.

Figure 5 demonstrates variation of nondimensionalized entropy jumps across a normal steady shock as a function of the upstream Mach numbers for ratios of secondary to shear viscosity λ/μ = -2.0 (dash-double-dot line), λ/μ = - 1.3333 (dash-dot line), λ/μ = - 0.6666 (solid line), λ/μ = 0.0 (long-dashed line), λ/μ = 0.6666 (dotted line), and λ/μ = 1.3333 (dashed line). Notice that Rankine-Hugoniot conditions correspond to the results with λ/μ = - 0.6666 (solid line) only.

Conclusions

It has been demonstrated that only if Stokes hypothesis is used (postulating that bulk viscosity is zero) the jump conditions across normal steady compression shocks will have the magnitudes given by the Rankine-Hugoniot jump conditions. For negative values of the bulk viscosity the shock strength will increase. If the bulk viscosity is positive the shocks will become weaker. It was also demonstrated that entropy reaches its maximum approximately at the middle of the shock and then rapidly decreases towards its considerably lower jump value after the shock.

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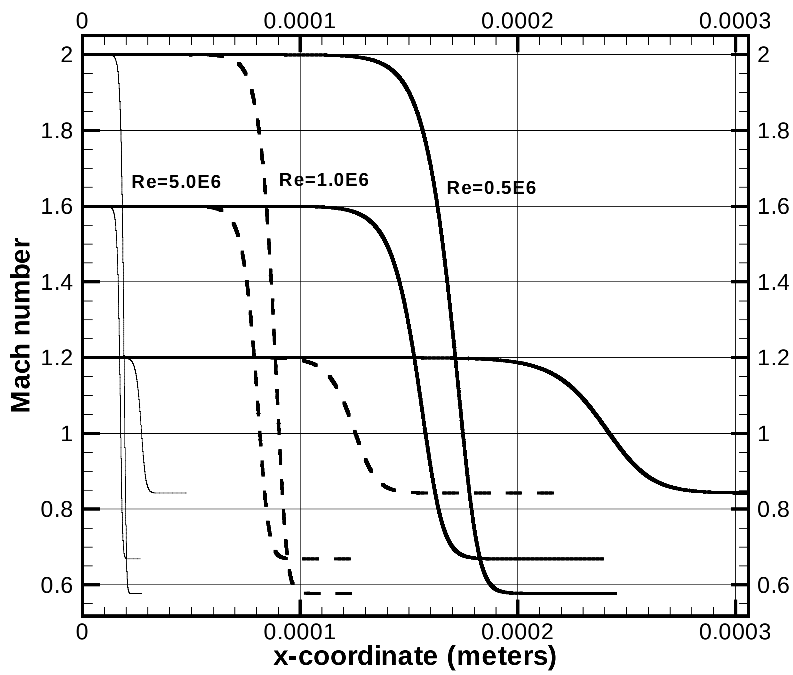

Fig. 1. Variation of Mach number through the normal steady shock for ratio of secondary to shear viscosity λ/μ = -0.6666, three values of upstream Mach number (M1 = 1.2, 1.6 and 2.0), and three values of Reynolds number (thick line Re = 500,000, dashed line Re = 1,000,000, and thin line Re = 5,000,000).

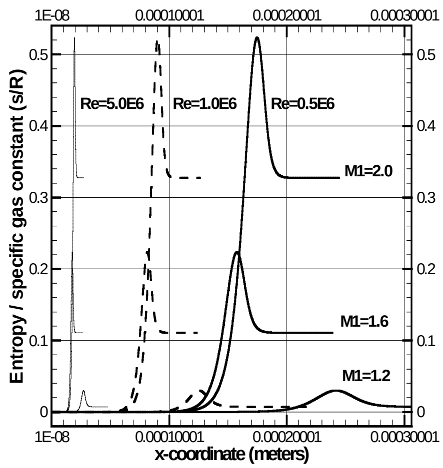

Fig. 2. Variation of nondimensionalized entropy through the normal steady shock for ratio of secondary to shear viscosity λ/μ = -0.6666, three values of upstream Mach number (M1 = 1.2, 1.6 and 2.0), and three values of Reynolds number (thick line Re = 500,000, dashed line Re = 1,000,000, and thin line Re = 5,000,000).

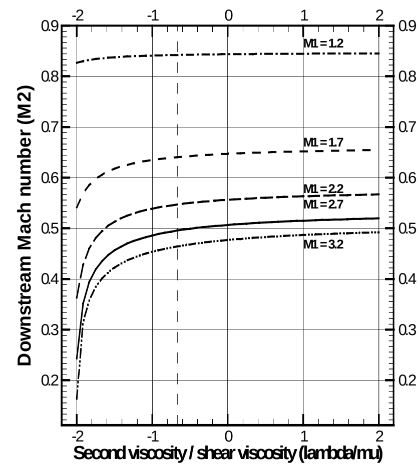

Fig. 3. Variation of Mach number downstream of a normal shock as a function of the ratios of secondary to shear viscosity λ/μ for five values of upstream Mach number M1. Notice that Rankine-Hugoniot conditions correspond to the results with λ/μ = -0.6666.

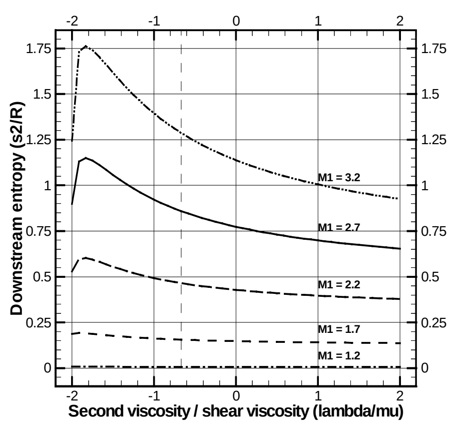

Fig. 4. Variation of nondimensionalized entropy downstream of a normal shock as a function of the ratios of secondary to shear viscosity λ/μ for five values of upstream Mach number M1. Notice that Rankine-Hugoniot conditions correspond to the results with λ/μ = - 0.6666.

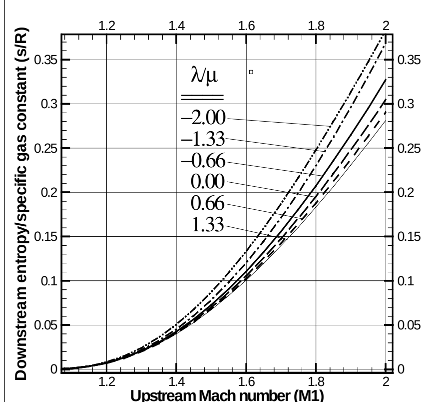

Fig. 5. Variation of nondimensionalized entropy downstream of a normal steady shock as a function of the upstream Mach numbers for six ratios of secondary to shear viscosity λ/μ. Notice that Rankine-Hugoniot conditions correspond to the results with λ/μ = - 0.6666 (solid line) only.