AMD-Vol. 244
MD-Vol. 92

RECENT ADVANCES IN THE MECHANICS OF STRUCTURED CONTINUA - 2000 -

Presented at

THE 2000 ASME INTERNATIONAL MECHANICAL ENGINEERING CONGRESS AND EXPOSITION
NOVEMBER 5–10, 2000
ORLANDO, FLORIDA

Sponsored by

THE APPLIED MECHANICS DIVISION, ASME
THE MATERIALS DIVISION, ASME

Edited by

MEHRDAD MASSOUDI
U.S. DEPARTMENT OF ENERGY

K. R. RAJAGOPAL
TEXAS A&M UNIVERSITY

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
Three Park Avenue / New York, N.Y. 10016

AMD-Vol. 244/MD-Vol. 92, Recent Advances in the Mechanics
of Structured Continua — 2000
ASME 2000

ELECTROMAGNETOHYDRODYNAMICS (EMHD): NUMERICAL EXPERIMENTS IN STEADY PLANAR FLOWS

Brian H. Dennis *
Department of Aerospace Engineering
233 Hammond Building
The Pennsylvania State University
University Park, PA 16802
Email: bhd102@psu.edu

George S. Dulikravich †
Department of Mechanical and Aerospace Engineering
UTA Box 19018
The University of Texas at Arlington
Arlington, TX 76019
Email: gsd@uta.edu

ABSTRACT

A new second order theoretical model of the combined interaction of externally applied electric and magnetic fields and viscous incompressible fluid flows has been rewritten as a system of first order partial differential equations. The system was solved using least-squares finite element model on an unstructured grid. The numerical algorithm is very stable and accurate. The accuracy was confirmed by comparing the numerical results against simple analytical results for a magnetohydrodynamic flow between two parallel isothermal infinite stationary plates. Effects of Joule heating and reverse pressure gradient are correctly predicted with this algorithm which is applicable to arbitrary planar flow configurations.

NOMENCLATURE

$\underline{B} = \mu_0(\underline{H} + \underline{M})$ magnetic flux density, $kg\ A^{-1}\ s^{-2}$
C_p specific heat at constant pressure, $K^{-1}\ m^2\ s^{-2}$
$\underline{\underline{d}} = \frac{1}{2}[\nabla\underline{v} + (\nabla\underline{v})^T]$ average rate of deformation tensor, s^{-1}
$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$ material derivative, s^{-1}
$\underline{D} = \varepsilon_0\underline{E} + \underline{P}$ electric displacement vector, $A\ s\ m^{-2}$
$\underline{E}$ electric field intensity, $kg\ m\ s^{-3}\ A^{-1}$
$\hat{\underline{E}} = \underline{E} + \underline{v}\times\underline{B}$ electromotive intensity, $kg\ m\ s^{-3}\ A^{-1}$
$\underline{H}$ magnetic field intensity, $A\ m^{-1}$
$\underline{\underline{I}}$ unit tensor
$\underline{J} = \underline{J_c} + q_e\underline{v}$ electric current density, $A\ m^{-2}$
$\underline{J_c}$ electric conduction current, $A\ m^{-2}$
$\underline{M}$ total magnetization per unit volume, $A\ m^{-1}$
$\hat{\underline{M}} = \underline{M} + \underline{v}\times\underline{P}$ magnetomotive intensity, $A\ m^{-1}$
p pressure, $kg\ m^{-}1\ s^{-2}$
$\underline{P}$ total polarization per unit volume, $A\ s\ m^{-2}$
q_e total electric charge per unit volume, $A\ s\ m^{-3}$
$\underline{q}$ conduction heat flux, $kg\ s^{-3}$
Q_h heat source per unit volume, $kg\ m^{-1}\ s^{-3}$
t time, s
$\underline{\underline{t}} = -\varphi\underline{\underline{I}} + \underline{\underline{\tau}}$ Cauchy stress tensor, $kg\ m^{-1}\ s^{-2}$
T absolute temperature, K
$\hat{u}$ internal energy per unit mass, $m^2\ s^{-2}$
$\underline{v}$ fluid velocity, $m\ s^{-1}$

Greek Symbols

ε electric permittivity, $kg^{-1}\ m^{-3}\ s^4\ A^2$
ε_0 electric permittivity of vacuum, $kg^{-1}\ m^{-3}\ s^4\ A^2$
$\varepsilon_p = \varepsilon_0\chi_e$ polarization electric permittivity, $kg^{-1}\ m^{-3}\ s^4\ A^2$
$\varepsilon_r = \varepsilon/\varepsilon_0$ relative electric permittivity
η fluid viscosity, $kgm^{-1}s^{-1}$
ϕ electric potential, Vm
φ modified hydrostatic pressure, $kgm^{-1}s^{-2}$
ρ fluid density, $kg\ m^{-3}$
$\underline{\underline{\tau}}$ deviator part of stress tensor, $kg\ m^{-1}\ s^{-2}$
μ magnetic permeability, $kg\ m\ A^{-2}\ s^{-2}$
$\mu_0 = 4\pi\times10^{-7}$ magnetic permeability of vacuum, $kgmA^{-2}s^{-2}$
$\mu_r = \mu/\mu_0$ relative magnetic permeability
$\mu_m = \mu_0/\chi_B$ magnetization magnetic permeability, $kgmA^{-2}s^{-2}$
$\chi_B = 1 - \mu_r^{-1}$ magnetic susceptibility based on B

*Graduate Research Assistant. Student member ASME.

$\chi_e = \varepsilon_r - 1$ electric susceptibility
$\underline{\omega}$ vorticity, s^{-1}

INTRODUCTION

The study of fluid flows containing electric charges under the influence of an externally applied electric field and negligible magnetic field are known as electrohydrodynamics or EHD (Melcher, 1982). The study of fluid flows influenced only by an externally applied magnetic field without electric charges in the fluid is known as magnetohydrodynamics or MHD (Sutton and Sherman, 1965). Numerous publications are available dealing with the EHD and the MHD models, their numerical simulations, and applications (Dulikravich, 1999; Dulikravich and Ahuja, 1993; Dulikravich, Ahuja and Lee, 1993; 1994; Dulikravich, Choi, and Lee, 1994; Dennis and Dulikravich, 2000a; 2000b). Although fairly complex, the existing mathematical models for EHD and MHD often represent unacceptably simplified and inconsistent model of the actual physics (Landau and Lifshitz, 1960; Stuetzer, 1962; Pai, 1963; Dulikravich and Lynn, 1997b). The study of fluid flows under the combined influence of the externally applied and internally generated electric and magnetic fields is often called electromagnetofluiddynamics (EMFD) (Hughes and Young, 1966; Eringen and Maugin, 1990a; 1990b; Rajagopal and Ruzicka, 1996; Dulikravich and Lynn, 1997a; 1997b). However, mathematical model for such combined electromagnetic field interaction with fluid flows is extremely complex and requires a large number of new physical properties of the fluid that cannot be found in the open literature. Thus, a somewhat simplified mathematical model should be used in actual numerical simulations of fluid flows under the combined influence of the externally applied electric and magnetic fields. In the case of incompressible fluids, such a non-linear model termed electromagnetohydrodynamics (EMHD) was recently derived by Ko and Dulikravich (1998 and 2000; 1999a and 2000a; 1999b and 2000b). This is a second order theory that is fully consistent with all of the basic assumptions of the complete EMFD model. The basic assumptions are that the electric and magnetic fields, rate of strain, and temperature gradient are relatively small. Furthermore, terms of second order and higher in the average rate of deformation tensor are neglected as in the case of conventional Newtonian fluids. Only the terms up to second order in $\underline{\underline{d}}, \underline{\hat{E}}, \underline{B}$, ∇T are retained. Because of the unavailability of the complete EMHD model until recently and because of the considerable complexity of even simpler versions of the EMHD model, it is still hard to find publications dealing with the combined influence of electric and magnetic fields and the fluid flow (Gerbeth, Thess, and Marty, 1990). The objective of this paper is to present numerical results for the complete second-order theory EMHD model in the case of two-dimensional planar flows. The numerical results will be presented for steady laminar flows of homocompositional Newtonian fluids. The numerical results will also be verified against simple analytical solutions.

SECOND ORDER ANALYTICAL MODEL OF EMHD

A full system of partial differential equations governing incompressible flows under the combined effects of electromagnetic forces (Ko and Dulikravich, 1998 and 2000; 1999a and 2000a; 1999b and 2000b) is summarized in this section by using the constitutive equations which have been derived through the second order theory. Specifically, polarization and magnetization vectors are defined as

$$\underline{P} = \varepsilon_0 \chi_e \underline{\hat{E}} \equiv \varepsilon_p \underline{\hat{E}}, \quad \underline{\hat{M}} = \frac{\chi_B}{\mu_0} \underline{B} \equiv \frac{\underline{B}}{\mu_m}. \tag{1}$$

which indicates a medium with purely instantaneous response (Lakhtakia and Weiglhofer, 1995). The deviator part of the stress tensor is defined as

$$\begin{aligned}\underline{\underline{\tau}} = 2\mu_v \underline{\underline{d}} - \sigma_2 \underline{\hat{E}} \otimes \underline{\hat{E}} - T^{-1}\kappa_2 \nabla T \otimes \nabla T \\ -(T^{-1}\kappa_5 + \sigma_5)(\underline{\hat{E}} \cdot \nabla T)_S,\end{aligned} \tag{2}$$

Electric current conduction vector is defined as

$$\begin{aligned}\underline{J_c} = \sigma_1 \underline{\hat{E}} + \sigma_2 \underline{\underline{d}} \cdot \underline{\hat{E}} + \sigma_4 \nabla T + \sigma_5 \underline{\underline{d}} \cdot \nabla T \\ + \sigma_7 \underline{\hat{E}} \times \underline{B} + T^{-1}\kappa_8 \nabla T \times \underline{B},\end{aligned} \tag{3}$$

while thermal conduction flux is defined as

$$\underline{q} = \kappa_1 \nabla T + \kappa_2 \underline{\underline{d}} \cdot \nabla T + \kappa_4 \underline{\hat{E}} + \kappa_7 \nabla \times \underline{B} + \kappa_8 \underline{\hat{E}} \times \underline{B}, \tag{4}$$

Then, Maxwell's equations become

$$\nabla \cdot \left(\varepsilon_0 \underline{E} + \varepsilon_p \underline{\hat{E}}\right) = q_e, \tag{5}$$

$$\nabla \cdot \underline{B} = 0, \tag{6}$$

$$\nabla \times \underline{E} = \frac{\partial \underline{B}}{\partial t}, \tag{7}$$

$$\begin{aligned}\nabla \times \left(\frac{\underline{B}}{\mu} + \varepsilon_p \underline{v} \times \underline{\hat{E}}\right) = \frac{\partial}{\partial t}\left(\varepsilon_0 \underline{E} + \varepsilon_p \underline{\hat{E}}\right) \\ + q_e \underline{v} + \sigma_1 \underline{\hat{E}} + \sigma_2 \underline{\underline{d}} \cdot \underline{\hat{E}} + \sigma_4 \nabla T \\ + \sigma_5 \underline{\underline{d}} \cdot \nabla T + \sigma_7 \underline{\hat{E}} \times \underline{B} + T^{-1}\kappa_8 \nabla T \times \underline{B}.\end{aligned} \tag{8}$$

while the Navier-Stokes equations become

$$\nabla \cdot \underline{v} = 0, \tag{9}$$

$$\rho \frac{D\underline{v}}{Dt} = -\rho g[1 - \alpha(T - T_0)]\underline{i_3} - \nabla(p + p_e + p_m)$$

$$
\begin{aligned}
&+ \nabla\cdot\left(\mu_v\left(\nabla \underline{v}+\nabla \underline{v}^T\right)\right) - \nabla\cdot\left(\sigma_2(\underline{\hat{E}}\otimes\underline{\hat{E}})\right)\\
&- \nabla\cdot\left(T^{-1}\kappa_2\left(\nabla T\otimes\nabla T\right)\right) + q_e\underline{\hat{E}}\\
&- \nabla\cdot\left((T^{-1}\kappa_5+\sigma_5)(\underline{\hat{E}}\otimes\nabla T)_S\right) + \sigma_1\underline{\hat{E}}\times\underline{B} \qquad (10)\\
&+ \sigma_2\underline{\underline{d}}\cdot\underline{\hat{E}}\times\underline{B} + \sigma_4\nabla T\times\underline{B} + \sigma_5\underline{\underline{d}}\cdot\nabla T\times\underline{B}\\
&+ \sigma_7(\underline{\hat{E}}\times\underline{B})\times\underline{B} + T^{-1}\kappa_8(\nabla T\times\underline{B})\times\underline{B}\\
&+ \varepsilon_p(\nabla\underline{E})\cdot\underline{\hat{E}} + (\nabla\underline{B})\cdot\left(\frac{\underline{B}}{\mu_m} - \varepsilon_p\underline{v}\times\underline{\hat{E}} + \frac{D}{Dt}(\varepsilon_p\underline{\hat{E}}\times\underline{B})\right),
\end{aligned}
$$

$$
\begin{aligned}
\rho C_p\frac{DT}{Dt} &= Q_h + \nabla\cdot\left(\kappa_1\nabla T + \kappa_2\underline{\underline{d}}\cdot\nabla T + \kappa_4\underline{\hat{E}} + \kappa_5\underline{\underline{d}}\cdot\underline{E}\right.\\
&\left.+ \kappa_7\nabla T\times\underline{B} + \kappa_8\underline{E}\cdot\underline{B}\right) + \sigma_1\underline{\hat{E}}\cdot\underline{\hat{E}} + \sigma_4\underline{\hat{E}}\cdot\nabla T \qquad (11)\\
&- \frac{\kappa_2}{T}\nabla T\cdot\underline{\underline{d}}\cdot\nabla T - \frac{\kappa_5}{T}\underline{\hat{E}}\cdot\underline{\underline{d}}\cdot\nabla T + \frac{\kappa_8}{T}\underline{\hat{E}}\cdot(\nabla T\times\underline{B})\\
&+ +\underline{\hat{E}}\cdot\frac{D(\varepsilon\underline{\hat{E}})}{Dt} - \frac{\underline{B}}{\mu_m}\cdot\frac{D\underline{B}}{Dt}.
\end{aligned}
$$

Notice that in this the EMHD model the physical properties of the incompressible fluid, $\chi_e, \chi_B, \mu_v, \sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7, \kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_8, \alpha$, can be either constants or functions of temperature only.

LEAST-SQUARES FINITE ELEMENT METHOD

The system of partial differential equations described in section is discretized using the least squares finite element method (LSFEM). We first look at the LSFEM for a general linear first-order system (Jiang,1992)

$$\underline{\underline{L}}\underline{u} = \underline{f} \qquad (12)$$

where

$$\underline{\underline{L}} = \underline{\underline{A_1}}\frac{\partial}{\partial x} + \underline{\underline{A_2}}\frac{\partial}{\partial y} + \underline{\underline{A_3}} \qquad (13)$$

for two-dimensional problems. The residual of the system is represented by $\underline{R}$.

$$\underline{R}(\underline{u}) = \underline{\underline{L}}\underline{u} - \underline{f} \qquad (14)$$

We now define the following least squares functional I over the domain Ω

$$I(\underline{u}) = \int_\Omega \underline{R}(\underline{u})^T\cdot\underline{R}(\underline{u})\,dx\,dy \qquad (15)$$

The weak statement is then obtained by taking the variation of I with respect to $\underline{u}$ and setting the result equal to zero.

$$\delta I(\underline{u}) = \int_\Omega \left(\underline{\underline{L}}\delta\underline{u}\right)\left(\underline{\underline{L}}\underline{u} - \underline{f}\right)dx\,dy = 0 \qquad (16)$$

Using equal order shape functions, $\underline{\hat{\phi}}_i$, for all unknowns, the vector $\underline{u}$ is written as

$$\underline{u} = \sum_{i=1}^{n}\underline{\hat{\phi}}_i\{u_1,u_2,u_3,\ldots,u_m\}_i^T \qquad (17)$$

where $\{u_1,u_2,u_3,\ldots,u_m\}_i$ are the nodal values at the ith node of the finite element. Introducing the above approximation for $\underline{u}$ into the weak statement leads to a linear system of algebraic equations

$$\underline{\underline{K}}\underline{U} = \underline{F} \qquad (18)$$

where $\underline{\underline{K}}$ is the stiffness matrix, $\underline{U}$ is the vector of unknowns, and $\underline{F}$ is the force vector.

NONDIMENSIONAL FIRST ORDER FORM FOR SIMPLIFIED EMHD

The full system of partial differential equations describing EMHD flows contain many parameters that refer to physical properties that are not known at this time. Rather than complete numerical simulations with guessed values of this parameters, we chose work with only those terms for which the material properties are known. In this case, we simplify the equations by retaining only source terms that contain κ_1 and σ_1 since these values are available for various fluids.

Use of LSFEM for systems of equations that contain higher order derivatives is usually difficult due to the higher continuity restrictions imposed on the approximation functions. For this reason it is more convenient to transform the system into an equivalent first order form before applying LSFEM. For the case of electromagnetohydrodynamics, the second order derivatives are transformed by introducing vorticity, $\underline{\omega}$, as an additional unknown. The energy equation is also transformed into first order form by introducing heat fluxes as additional unknowns.

$$
\begin{aligned}
\nabla\cdot\underline{v}^* &= 0 \qquad (19)\\
\underline{v}^*\cdot\nabla\underline{v}^* + \tfrac{1}{Re}\nabla\times\underline{\omega}^* + \nabla p^* - \tfrac{Ht^2}{Re}\underline{v}^*\times\underline{B}^*\times\underline{B}^* &\\
-Seq_e^*\underline{E}^* - M_1\underline{E}^*\times\underline{B}^* &= 0 \qquad (20)\\
\underline{\omega}^* - \nabla\times\underline{v}^* &= 0 \qquad (21)\\
\underline{v}^*\cdot\nabla T^* + \nabla\cdot\underline{q}^* - \tfrac{Ht^2Ec}{Re}(\underline{v}^*\times\underline{B}^*)^2 + E_1\underline{E}^2 &= 0 \qquad (22)\\
\underline{q}^* + \tfrac{1}{Pe}\nabla T^* &= 0 \qquad (23)\\
\nabla\times\underline{q}^* &= 0 \qquad (24)\\
\nabla\cdot\underline{B}^* &= 0 \qquad (25)\\
\nabla\times\underline{B}^* = Rm\underline{v}^*\times\underline{B}^* + B_1\underline{v}^*q_e^* + B_2\underline{E}^* & \qquad (26)
\end{aligned}
$$

$$\nabla \cdot \underline{E}^* = Neq_e^* \quad (27)$$

$$\nabla \times \underline{E}^* = 0 \quad (28)$$

$$\nabla \phi^* = \underline{E}^* \quad (29)$$

where $\underline{v}^* = \underline{V} v_0^{-1}$, $\underline{B}^* = \underline{B} B_0^{-1}$, $\underline{E}^* = \underline{E} E_0^{-1}$,$q_e^* = q_e q_{e_0}$,$E_0^{-1}$, $p^* = p\rho^{-1} v_0^{-2}$, $x^* = x L_0^{-1}$, $y^* = y L_0^{-1}$, $T^* = \frac{T - T_{cold}}{\Delta T_0}$. Here, L_0 is the reference length, v_0 is the reference speed, and B_0 is the reference magnetic flux density. The temperature is nondimensionalized with a temperature difference, ΔT_0, where $\Delta T_0 = T_{hot} - T_{cold}$. For convenience the $*$ superscript will be dropped for the remainder of the paper.

The nondimensional numbers are given by:

$$\begin{array}{lll} Ne = \frac{q_{e_0} L_0^2}{\varepsilon_0 \Delta\phi_0}, & Rm = \mu\sigma v_0 L_0, & B_1 = \frac{L\mu_0 v_0 q_{e_0}}{B_0}, \\ B_2 = \frac{\mu_0 \sigma_0 \Delta\phi_0}{B_0}, & Re = \frac{\rho_0 v_0 L_0}{\eta_0}, & Ht = L_0 B_0 \sqrt{\frac{\sigma_0}{\eta_0}}, \\ Se = \frac{q_{e_0} \Delta\phi}{\rho v_0^2}, & M_1 = \frac{\Delta\phi_0 B_0 \sigma_0}{\rho_0 v_0^2}, & Pe = \frac{L_0 v_0 \rho_0 C p_0}{k_0}, \\ Ec = \frac{v_0^2}{C p_0 \Delta T_0}, & E_1 = \frac{\sigma_0 \Delta\phi_0^2}{v_0 \rho_0 C_{p_0} L_0} & \end{array} \quad (30)$$

It should be noted that a curl free condition on the heat flux vector field appears in the first-order form of energy equation. It was shown in Jiang and Povinelli (1993) that the presence of this condition is required for achieving optimal convergence rates for the heat flux vector, $\underline{q}$. It was also shown in Jiang and Povinelli that the inclusion of the curl free condition does not produce an over determined system of equations. For the electric field equations, the first order form of Maxwell's equations does not include electric potential. Since the most common boundary conditions for static electric fields are given in terms of potential, it is necessary to add the equation (27) for electric potential, ϕ.

We now write the above system in the general form of a first-order system (12). Although the entire system written in (19)-(29) can be treated by LSFEM, it was found to be more economical to solve the fluid, heat transfer, and electromagnetic field equations separately, in an iterative manner. Here, a general form first order system is written for the fluid system (19)- (21) and denoted by the superscript *fluid*. A first-order system is also written in general form for the electromagnetic field equations (25)-(29) and is denoted by the superscript *em*. The first-order system written in general form for the heat transfer equations (22)-(24) is denoted by the superscript *heat*. In addition, the nonlinear convective terms in the fluid equations are linearized with Newton's method leading to a system suitable for treatment with the LSFEM. For the two-dimensional case we specify the z component of the magnetic field and assume the x and y components are zero. The x and y components for velocity, $\underline{v}$, and electric field, $\underline{E}$, are left as unknowns while their z components are assumed to be zero. For simplicity, we consider flows that do not contain free charged particles.

$$\underline{\underline{A}}_1^{fluid} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_0 & 0 & 1 & 0 \\ 0 & u_0 & 0 & -\frac{1}{Re} \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \underline{\underline{A}}_2^{fluid} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ v_0 & 0 & 0 & \frac{1}{Re} \\ 0 & v_0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\underline{\underline{A}}_3^{fluid} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{Ht^2}{Re} B_{z0}^2 + \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} & 0 & 0 \\ \frac{\partial v_0}{\partial x} & \frac{Ht^2}{Re} B_{z0}^2 + \frac{\partial v_0}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\underline{f}^{fluid} = \begin{Bmatrix} 0 \\ u_0 \frac{\partial u_o}{\partial x} + v_0 \frac{\partial u_o}{\partial y} - M_1 E_{y_0} B_{z_0} \\ u_0 \frac{\partial v_o}{\partial x} + v_0 \frac{\partial v_o}{\partial y} + M_1 E_{x_0} B_{z_0} \\ 0 \end{Bmatrix}, \underline{u}^{fluid} = \begin{Bmatrix} u \\ v \\ p \\ \omega \end{Bmatrix} \quad (31)$$

$$\underline{\underline{A}}_1^{em} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \underline{\underline{A}}_2^{em} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\underline{\underline{A}}_3^{em} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{f}^{em} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \underline{u}^{em} = \begin{Bmatrix} \phi \\ E_x \\ E_y \end{Bmatrix} \quad (32)$$

$$\underline{\underline{A}}_1^{heat} = \begin{bmatrix} u_0 & 1 & 0 \\ \frac{1}{Pe} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \underline{\underline{A}}_2^{heat} = \begin{bmatrix} v_0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{Pe} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\underline{\underline{A}}_3^{heat} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{f}^{heat} = \begin{Bmatrix} E_1 (E_{x_0}^2 + E_{y_0}^2) + \frac{Ht^2 Ec}{Re} (v_0^2 B z_0^2 + u_0^2 B_{z_0}^2) \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\underline{u}^{heat} = \begin{Bmatrix} T \\ q_x \\ q_y \end{Bmatrix} \quad (33)$$

A solution satisfying all of the above systems of equations can be found by using a simple iterative process. First, the system given in (32) is solved for the electric field. The system in (31) is solved with the electric field and velocities from an initial guess or from the previous iteration. Here, quantities taken from the previous iteration are designated with the subscript 0. This process is repeated until a specified convergence tolerance is reached. For most cases considered in this paper, reduction of the residual norm of both systems by 3.5 orders of magnitude was achieved in less than 5 iterations. Once the velocity and electric fields are determined, the system in (33) is solved to obtain the temperature distribution.

VERIFICATION OF ACCURACY

It is difficult to verify the accuracy of an EMHD code. This is due to the absence of analytical solutions for such equations. However, analytical solutions do exist for MHD flows. Here we will use such an analytic solution to validate the MHD portion of the code.

The accuracy of the LSFEM for MHD was tested against etic solutions for Poisuille-Hartmann flow. The Poiseuille-Hartmann flow is a 1-D flow of a conducting and viscous fluid between two stationary plates with a uniform external magnetic field applied orthogonal to the plates. Assuming the walls are at $y = \pm L$ and that fluid velocity on the walls is zero and that the fluid moves in the x-direction under the influence of a constant pressure gradient, then the velocity profile is given by (Hughes and Young, 1966)

$$u(y) = \frac{\rho Ht}{\sigma B_y^2}\frac{\partial p}{\partial x}\left(\frac{\cosh(Ht) - \cosh(\frac{Hty}{L})}{\sinh(Ht)}\right) \tag{34}$$

The movement of the fluid induces a magnetic field in the x-direction and is given by

$$B_x(y) = \frac{B_y Rm}{Ht}\left(\frac{\sinh(\frac{Hty}{L}) - \frac{y}{L}\sinh(Ht)}{\cosh(Ht) - 1}\right) \tag{35}$$

A test case was run using the parameters given in Table 1 and with a mesh composed of 2718 parabolic triangular elements. Figure 1 shows the computed and analytical results for the velocity profile. Figure 2 shows the computed and analytical results for the induced magnetic field. For both cases, one can see that the agreement between the analytical solution and the LSFEM solution is excellent (Dennis and Dulikravich, 2000a).

NUMERICAL RESULTS

The LSFEM formulation for EMHD will now be demonstrated for a simple flow through a channel under the influence of externally applied electric and magnetic fields. The domain had a height of 4 cm and length of 40 cm. A parabolic velocity profile was specified at the inlet and a uniform pressure enforced at the exit. A no-slip condition for velocity and temperature were specified on the walls of the channel. A positive electrode was placed on the bottom wall and a negative electrode on the top wall. A potential difference of 50 volts was applied across them. A uniform magnetic field of 0.05 T was specified in the z direction on the entire domain. Other parameters relevant to this example are listed in Table 2. A mesh composed of 3422 parabolic triangular elements was used. A portion of the mesh can be seen in figure 3. Figure 4 shows the computed electric potential contours which are consistent with the exact solution. It can be seen from the velocity vectors in figure 5 that the flow is moving from the left to the right. However, figure 6 shows that the pressure is increasing from left to right. In this example, the pressure increases by $34.6Pa$ from inlet to outlet. A typical fully developed laminar channel flow with no applied electric and magnetic fields would show a uniform drop in pressure. In this example it is the interaction of the electric and magnetic fields with the moving fluid that is driving the flow. Another interesting effect of the applied electric and magnetic fields is the increase in temperature as the flow moves downstream as seen in figures 7 and 8. This is due to Joule heating and is a direct effect of the source terms in the energy equation involving electric and magnetic field terms.

CONCLUSION

A unified theoretical model of simultaneously applied and interacting electric and magnetic fields and incompressible homocompositional viscous fluid flows has been expressed as a coupled sequence of first order partial differential equation systems. These systems were discretized using a least-squares finite element method and integrated on a unstructured computational grid. Numerical results are in excellent agreement for the test case of a steady laminar flow between infinite parallel plates with simultaneously applied uniform vertical electric field and a uniform horizontal magnetic field. Joule heating effects are clearly predicted. This numerical algorithm and the accompanying software are applicable to arbitrary two-dimensional planar EMHD flow-field analyses.

ACKNOWLEDGMENT

The authors are grateful for the partial support by the ALCOA Foundation Grant facilitated by Dr. Yimin Ruan and Dr. Owen Richmond of ALCOA Technical Center, and for the partial support by Lawrence Livermore National Laboratory grant LLNL-B344847 monitored by Dr. J. Ray Smith and Dr. LouAnn Tong.

REFERENCES

Dennis, B. H., and Dulikravich, G. S., 2000a, "Optimization of Magneto-Hydrodynamic Control of Diffuser Flows Using Micro-Genetic Algorithm and Least Squares Finite Elements", Journal of Finite Elements in Analysis and Design, 2000.

Dennis, B. H., and Dulikravich, G. S., 2000b, "Simulation of Magnetohydrodynamics With Conjugate Heat Transfer", EC-COMAS 2000, European Congress on Computational Methods in Applied Sciences and Engineering, (ed: E. Onate), September 11-14, 2000, Barcelona, Spain.

Dulikravich, G. S., June 1999, "Electro-Magneto-Hydrodynamics and Solidification," Chapter no. 9 in Advances in Flow and Rheology of Non-Newtonian Fluids, Part B (eds: D. A. Siginer, D. De Kee and R. P. Chhabra), Rheology Series, 8, Elsevier Publishers, pp. 677-716.

Dulikravich, G. S., and Ahuja, V., 1993&1994, "Modeling Dielectric Fluids Solidification With Charged Particles in Electric Fields and Reduced Gravity", ASME WAM'93, Symposium on Fundamentals of Heat Transfer in a Microgravity Environment, (eds: C. T. Avedisian and V. Arpaci), New Orleans, LA, Nov. 29- Dec. 3, 1993; also in Numerical Heat Transfer, Fundamentals, Part B, Vol. 25, No. 3, pp. 357-373, 1994.

Dulikravich, G. S., Ahuja, V., and Lee, S., 1994, "Modeling Three-Dimensional Solidification With Magnetic Fields and Reduced Gravity", International Journal of Heat and Mass Transfer, Vol. 37, No. 5, pp. 837-853.

Dulikravich, G. S., Ahuja, V., and Lee, S., 1993, "Simulation of Electrohydrodynamic Enhancement of Laminar Flow Heat Transfer", ASME National Heat Transfer Conference, Symposium on Fundamentals of Heat Transfer in Electromagnetic, Electrostatic, and Acoustic Fields, (eds: Y. Bayazitoglu and V. S. Arpaci), Atlanta, GA, August 8-11, 1993, ASME HTD-Vol. 248, pp. 43-52; also in Journal of Enhanced Heat Transfer, Vol. 1, No. 1, pp. 115-128.

Dulikravich, G. S., Choi, K. - Y., and Lee, S., 1994, "Magnetic Field Control of Vorticity in Steady Incompressible Laminar Flows", Symposium on Developments in Electrorheological Flows and Measurement Uncertainty, ASME WAM'94, (eds: D. A. Siginer, J. H. Kim, S. A. Sheriff and H. W. Colleman), Chicago, IL, Nov. 6-11, 1994, ASME FED-Vol. 205/AMD-Vol. 190, pp. 125-142.

Dulikravich, G. S., and Lynn, S. R., September 1997, "Unified Electro-Magneto-Fluid Dynamics (EMFD): Introductory Concepts", International Journal of Non-Linear Mechanics, Vol. 32, No. 5, pp. 913-922.

Dulikravich, G. S., and Lynn, S. R., September 1997, "Unified Electro-Magneto-Fluid Dynamics (EMFD): A Survey of Mathematical Models", International Journal of Non-Linear Mechanics, Vol. 32, No. 5, pp. 923- 932.

Eringen, A.C., and Maugin, G.A., 1990a, Electrodynamics of Continua I; Foundations and Solid Media, Springer-Verlag, New York.

Eringen, A.C., and Maugin, G.A., 1990b, Electrodynamics of Continua II; Fluids and Complex Media, Springer-Verlag, New York.

Gerbeth, G., Thess, A., and Marty, P., 1990, "Theoretical Study of the MHD Flow Around a Cylinder in Crossed Electric and Magnetic Fields", European Journal of Mechanics, B/Fluids, Vol. 9, No. 3, pp. 239- 257.

Hughes, W. F., and Young, F. J., 1966, The Electromagnetodynamics of Fluids, John Wiley and Sons, New York.

B.-N. Jiang, B.-N., 1992, "A Least-squares finite element method for incompressible Navier-Stokes problems", International Journal for Numerical Methods in Fluids, Vol. 14, pp. 842-859.

Jiang, B.-N., and Povinelli, L.A., 1993,"Optimal least-square finite element method for elliptic problems",Comput. Methods Appl. Mech. Engrg., Vol. 102, pp. 199-212.

Ko, H.-J., and Dulikravich, G. S., 1998&2000, "A Fully Non-Linear Model of Electro-Magneto- Hydrodynamics", Symposium on Rheology and Fluid Mechanics of Non-Linear Materials, (eds: D. A. Siginer and D. De Kee), Anaheim, CA, Nov. 15-20, 1998, ASME FED-Vol. 246/MD-Vol. 81, pp. 173-182; also in Int. Journal of Non-Linear Mechanics, Vol. 35, No. 4, pp. 709-719.

Ko, H.-J., and Dulikravich, G. S., 1999a&2000a, "Non-Reflective Boundary Conditions for a Consistent Two-dimensional Planar Electro-Magneto-Hydrodynamic Flow Model," Forum on Functional Fluids, 1999 Joint ASME/JSME Fluids Engineering Conference, (eds: G. S. Dulikravich, S. Sawada, K. Yamane), San Francisco, CA, July 18 - 23, 1999, also in Int. Journal of Non-Linear Mechanics, Vol. 36, No. 1, 2000, pp. 155-163.

Ko, H.-J., and Dulikravich, G. S., 1999b&2000b, "Non-Reflective Boundary Conditions for a Consistent Model of Axisymmetric Electro-Magneto-Hydrodynamic Flows," Symposium on Rheology and Fluid Mechanics of Nonlinear Materials, ASME IMECE'99, (ed: D. A. Siginer), Nashville, TN, November 14-19, 1999, ASME FED- Vol. 249, pp. 97-104; also in Int. Journal of Nonlinear Sciences and Numerical Simulation, Vol. 1, No. 4, 2000.

Lakhtakia, A., and Weiglhofer, W. S., 1995, "On the Causal Constitutive Relations of Magnetoelectric Media", 1995 IEEE International Symposium on Electromagnetic Compatibility, Atlanta, GA, August 14-18, pp. 611-613.

Melcher, J. R., 1981, Continuum Electromechanics, MIT Press, Cambridge, MA. Landau, L. D., and Lifshitz, E. M., 1960, Electrodynamics of Continuous Media, Pergamon Press, New York. Pai, S.-I., 1963, Magnetogasdynamics and Plasma Dynamics, Springer-Verlag, Vienna, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Rajagopal, K. R., and Ruzichka, M., 1996, "On the modeling of electrorheological materials," Mechanics Res. Communications, Vol. 23, pp. 401-407.

Stuetzer, O.M., 1962, "Magnetohydrodynamics and Electrohydrodynamics," Physics of Fluids, Vol. 5, no. 5, pp. 534-544.

Sutton, G.W., and Sherman, A., 1965. Engineering Magnetohydrodynamics, McGraw Hill, New York.

Table 1. PARAMETERS FOR POISUILLE-HARTMANN FLOW TEST PROBLEM

Ht	10.0
Rm	6×10^{-7}
$L_0\,(m)$	1.0
$v_0\,(m s^{-1})$	0.6
$\eta\,(kg\,m^{-1}s^{-1})$	0.01
$B_0\,(T)$	1.0
$\mu\,(H m^{-1})$	1×10^{-6}
$\partial p/\partial x\,(Pa m^{-1})$	0.6
$\sigma\,(\Omega^{-1}m^{-1})$	1.0

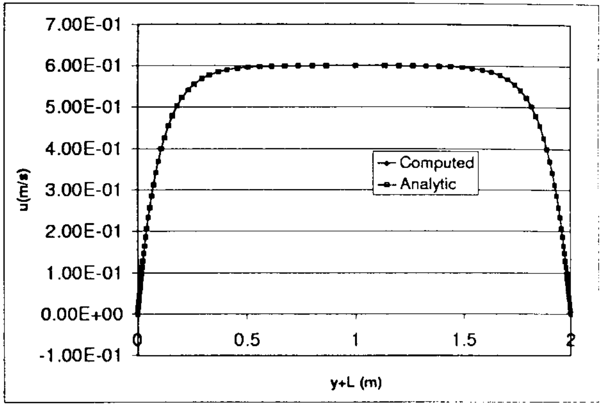
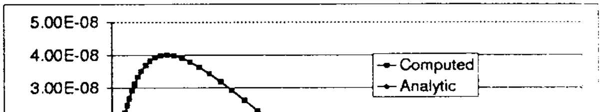


Figure 1. COMPUTED AND ANALYTICAL VALUES FOR VELOCITY PROFILE

Table 2. PARAMETERS FOR EMHD CHANNEL FLOW PROBLEM

$\rho(kgm^{-2})$	1055.0
Inlet height (cm)	4.0
Length (cm)	40.0
Inlet temperature (K)	310.0
Wall temperature (K)	298.0
$k(Wkg^{-1}K^{-1})$	.51
$C_p(Jkg^{-1}K^{-1})$	4178.0
inlet velocity (ms^{-1})	0.05
$\eta(kgm^{-1}s^{-1})$	0.004
[illegible]	1.4



outlet pressure (Pa)	1.0

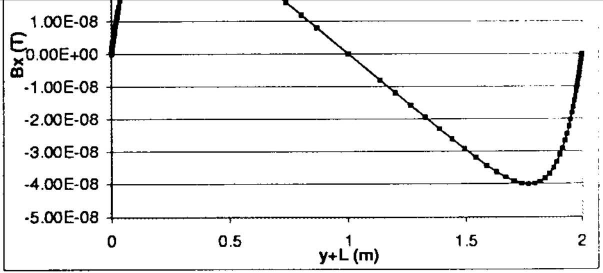


Figure 2. COMPUTED AND ANALYTICAL VALUES FOR INDUCED MAGNETIC FIELD

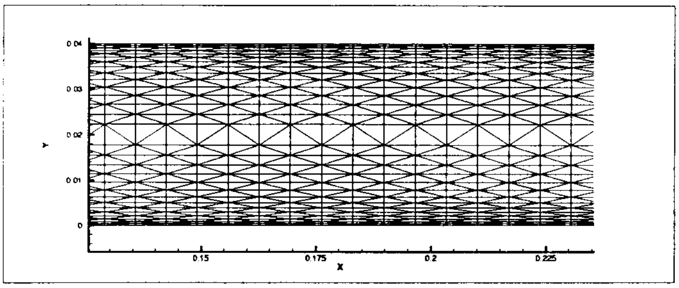


Figure 3. PORTION OF TRIANGULAR MESH USED FOR EMHD CHANNEL FLOW

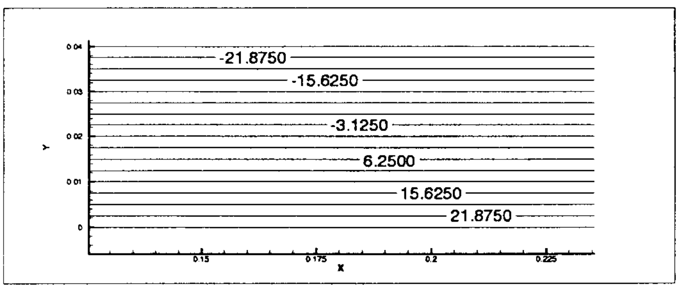


Figure 4. COMPUTED ELECTRIC POTENTIAL

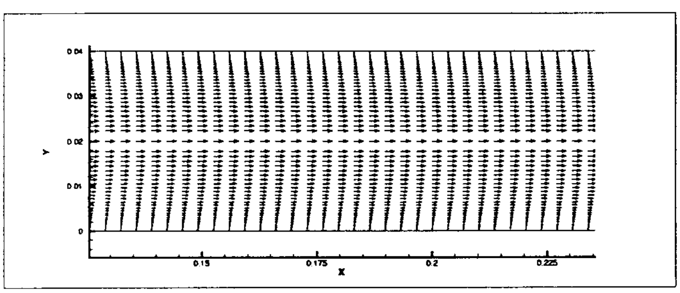


Figure 5. COMPUTED VELOCITY VECTORS

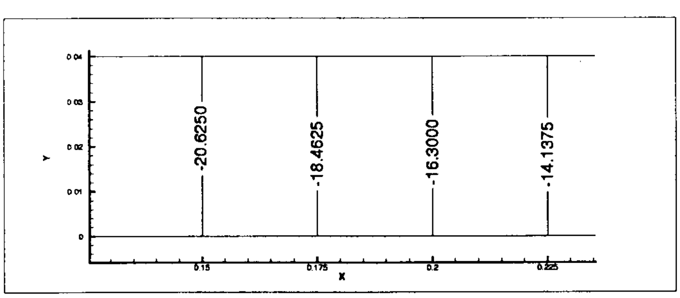


Figure 6. COMPUTED PRESSURE CONTOURS

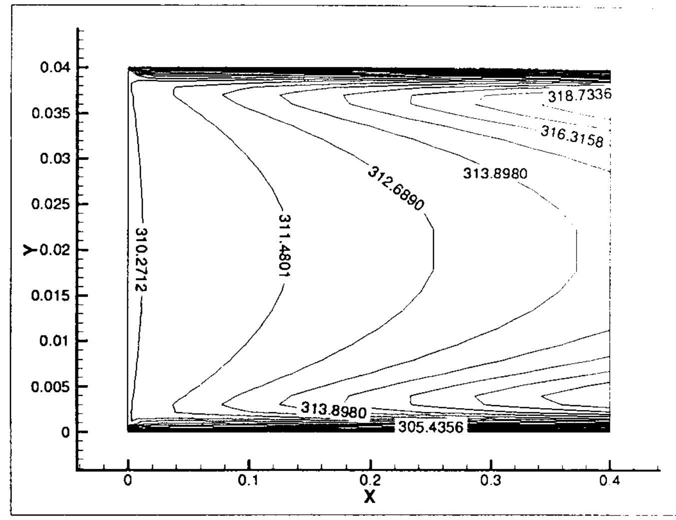


Figure 7. COMPUTED TEMPERATURE CONTOURS

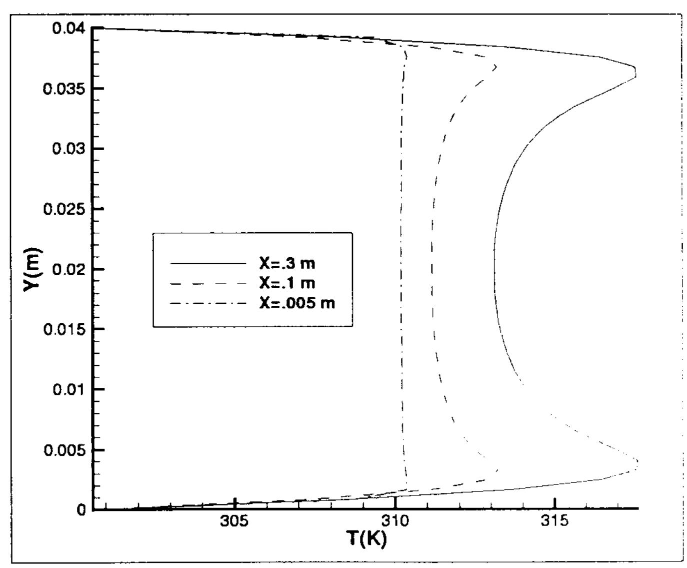


Figure 8. COMPUTED TEMPERATURE PROFILES