

**NON-REFLECTIVE BOUNDARY CONDITIONS FOR A
CONSISTENT TWO-DIMENSIONAL PLANAR
ELECTRO- MAGNETO-HYDRODYNAMIC FLOW MODEL**

Hyung-Jong Ko¹

Kumoh National University of Technology, Department of Mechanical Engineering
188 Shinpyung, Kumi, Kyungbuk, 730-701, Republic of Korea
phone: +82 (546) 467-4208 fax: +82 (546) 467-4200 e-mail: kohj@knut.kumoh.ac.kr

George S. Dulikravich²

The Pennsylvania State University, Department of Aerospace Engineering
University Park, Pennsylvania 16802, U. S. A.
phone: +1 (814) 863-0134 fax: +1 (814) 865-7092 e-mail: FT7@PSU.EDU

ABSTRACT

The electro-magneto-hydrodynamics (EMHD) deals with the motion of electrically conducting incompressible fluids under the combined influence of externally applied and internally generated electric and magnetic fields. In this paper, the non-reflective boundary conditions for the EMHD flow simulation have been studied. The consistent EMHD model with linear constitutive relations and artificial compressibility is expressed as a standard form in Cartesian coordinates. After some simplifications, the resulting EMHD system comprised of Maxwell equations for the electromagnetic field and modified Navier-Stokes equations for the flow-field, is transformed to a characteristic form, and the non-reflective boundary conditions are derived. The results show the strong mutual interactions between the flow-field and the electromagnetic field. The limiting cases, including the conventional flow-field model and the electromagnetic field model in vacuum, are recoverable from these results. The non-reflective boundary condition formulation for EMHD can be readily extended to the general curvilinear coordinate system and made suitable for direct numerical implementation.

NOMENCLATURE

$\underline{B} = \mu_0(\underline{H} + \underline{M})$ magnetic flux density, kg A⁻¹ s⁻²
 C_p specific heat at constant pressure, K⁻¹ m² s⁻²
 $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$ electric displacement vector, A s m⁻²

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$ material derivative, s⁻¹
 \underline{E} electric field intensity, kg m s⁻³ A⁻¹
 $\hat{\underline{E}} = \underline{E} + \underline{v} \times \underline{B}$ electromotive intensity, kg m s⁻³ A⁻¹
 \underline{f} mechanical body force per unit mass, m s⁻²
 \underline{F}^{em} electromagnetic body force per unit volume, kg m⁻² s⁻²
 \underline{H} magnetic field intensity, A m⁻¹
 $\underline{J} = \underline{J}_c + q_e \underline{v}$ electric current density, A m⁻²
 \underline{J}_c electric conduction current, A m⁻²
 \underline{M} total magnetization per unit volume, A m⁻¹
 $\hat{\underline{M}} = \underline{M} + \underline{v} \times \underline{P}$ magnetomotive intensity, A m⁻¹
 p pressure, kg m⁻¹ s⁻²
 $\tilde{p} = p + \frac{\epsilon_p(\hat{\underline{E}} \cdot \hat{\underline{E}})}{2} + \frac{(\underline{B} \cdot \underline{B})}{2\mu_m}$ modified pressure, kg m⁻¹ s⁻²
 \underline{P} polarization per unit volume, A s m⁻²
 q_e electric charge density, A s m⁻³
 \underline{q} conduction heat flux, kg s⁻³
 Q_h heat source per unit volume, kg m⁻¹ s⁻³
 \underline{t} Cauchy stress tensor, kg m⁻¹ s⁻²
 T absolute temperature, K
 \underline{v} fluid velocity, m s⁻¹

¹ Associate Professor.

² Associate Professor. Fellow ASME.

Greek Symbols

β	Chorin's artificial compressibility, $\text{kg m}^{-1} \text{s}^{-2}$
ϵ	electric permittivity, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_0 = 8.854 \times 10^{-12}$	electric permittivity of vacuum, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_p = \epsilon - \epsilon_0$	polarization electric permittivity, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_r = \epsilon/\epsilon_0$	relative electric permittivity
η	coefficient of fluid viscosity, $\text{kg m}^{-1} \text{s}^{-1}$
ρ	fluid density, kg m^{-3}
μ	magnetic permeability, $\text{kg m A}^{-2} \text{s}^{-2}$
$\mu_0 = 4\pi \times 10^{-7}$	magnetic permeability of vacuum, $\text{kg m A}^{-2} \text{s}^{-2}$
$\mu_m = \frac{\mu_0 \mu}{\mu - \mu_0}$	magnetization magnetic permeability, $\text{kg m A}^{-2} \text{s}^{-2}$

INTRODUCTION

Electro-magneto-hydrodynamics (EMHD) is the study of the flow of polarizable and magnetizable incompressible fluids under the combined effects of electric and magnetic fields. Hence, the conventional electro-hydrodynamics (EHD) and magneto-hydrodynamics (MHD) may be considered special cases of EMHD. There has been a series of theoretical studies on the EMHD flows (Melcher, 1981; Eringen and Maugin, 1990a, 1990b; Lakhtakia, 1993; Dulikravich and Lynn, 1995a, 1995b, 1997a, 1997b; Dulikravich and Jing, 1996, 1997; Dulikravich, 1998; Ko and Dulikravich, 1998). Based on continuum mechanics, the firm foundations of the EMHD theory were formulated by Eringen and Maugin (1990a; 1990b). They developed the most complete and robust model for the balance laws and constitutive equations. Following their general formulation, Ko and Dulikravich (1998) developed a fully consistent non-linear multi-dimensional EMHD model.

In order to develop a computational code for the EMHD problems, it is essential to have a formulation of the complete boundary conditions including the open boundary conditions. Existing publications of numerical simulations of electromagnetic fields with fluid flow do not include the effects of the polarization and magnetization (Shankar et al., 1989; Shang, 1991). The analytical formulation of the EMHD performed by Dulikravich and Jing (1996) was inconsistent in accounting for the polarization and magnetization. Consequently, an attempt (Dulikravich and Jing, 1997) to derive non-reflective and characteristic boundary conditions for this EMHD model was also inconsistent. They also reviewed jump conditions at the solid wall or discontinuity surface.

There have been numerous studies on the characteristic and non-reflective boundary conditions. For example, these studies were performed in general hyperbolic systems of equations by Thompson (1987, 1990), in MHD by Sun et al. (1995), in acoustics by Reitsma et al. (1993), etc. In this paper, the formulation of the non-reflective boundary conditions at the open boundaries will be performed by following Thompson's approach. For this purpose, the consistent linear EMHD model will be investigated (Ko and Dulikravich, 1998). With the help of certain simplifications, the characteristic boundary conditions will be derived from the characteristic forms of the governing equations. This work will supplement the results of Dulikravich (1998) based on an inconsistent EMHD model.

GOVERNING EQUATIONS OF EMHD FLOWS

The full system of equations governing electro-magneto-hydrodynamic (EMHD) flows consists of Maxwell's equations governing electro-magnetism, the Navier-Stokes equations governing fluid flow, and constitutive equations (Wineman and Rajagopal, 1995) describing behavior of the fluid. Following the general electro-magneto-gas-dynamic (EMGD) theory of Eringen and Maugin (1990a; 1990b), the balance laws for the electro-magnetic field are represented by Maxwell's equations. Using a vector operator form, the Maxwell's equations can be written in the rationalized MKS system as follows:

$$\nabla \cdot \underline{D} = q_e, \quad (\text{Gauss' law}) \quad (1)$$

$$\nabla \cdot \underline{B} = 0, \quad (\text{conservation of magnetic flux}) \quad (2)$$

$$\frac{\partial \underline{D}}{\partial t} = \nabla \times \underline{H} - \underline{J}, \quad (\text{Ampere-Maxwell's law}) \quad (3)$$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}. \quad (\text{Faraday's law}) \quad (4)$$

The relations between flux density and field intensity vectors in the polarizable and magnetizable medium are

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}, \quad (5)$$

$$\underline{B} = \mu_0 (\underline{H} + \underline{M}). \quad (6)$$

The vectors of polarization and magnetization should be estimated from the constitutive equations.

Taking the divergence of Eqs. (3) and (4), and using Eqs. (1) and (2), we obtain the following equations.

$$\frac{\partial q_e}{\partial t} + \nabla \cdot \underline{J} = 0, \quad (7)$$

$$\frac{\partial}{\partial t} (\nabla \cdot \underline{B}) = 0. \quad (8)$$

Here the former is called the equation of charge conservation. The latter equation implies that if $\nabla \cdot \underline{B} \equiv 0$ at the initial time, then $\nabla \cdot \underline{B} \equiv 0$ for any later time, t . Therefore, the equation of conservation of magnetic flux (Eq. (2)) is required only to restrict the initial distribution of \underline{B} . We can dispense with this equation in the further analysis. The governing equations for the electromagnetic field consist of Eqs. (3), (4), and (7).

The balance laws of the thermo-mechanical field are comprised of the three conservation laws. The equations of conservation of mass,

linear momentum, and energy for the incompressible fluids can be expressed in a vector operator form as

$$\nabla \cdot \underline{v} = 0, \quad (9)$$

$$\rho \frac{D\underline{v}}{Dt} = \nabla \cdot \underline{\underline{t}} + \rho \underline{f} + \underline{F}^{em}, \quad (10)$$

$$\rho C_p \frac{DT}{Dt} = Q_h + \nabla \cdot \underline{q} + \underline{J}_c \cdot \hat{\underline{E}} + \hat{\underline{E}} \cdot \underline{M} \cdot \frac{D\underline{P}}{Dt} - \hat{\underline{M}} \cdot \frac{D\underline{B}}{Dt} \quad (11)$$

,respectively. Here,

$$\begin{aligned} \underline{F}^{em} = & q_e \underline{E} + \underline{J} \times \underline{B} + (\nabla \underline{E}) \cdot \underline{P} + (\nabla \underline{B}) \cdot \underline{M} \\ & + \nabla \cdot (\underline{v}(\underline{P} \times \underline{B})) + \frac{\partial}{\partial t} (\underline{P} \times \underline{B}) \end{aligned} \quad (12)$$

is the electromagnetic body force per unit volume (Eringen and Maugin, 1990a).

In order to completely determine the electro-magnetic and thermo-mechanical fields, the balance laws must be supplemented by the constitutive equations since the number of the balance equations is less than that of unknowns. The most general theory of constitutive equations determining the polarization, magnetization, conduction current, heat flux, and Cauchy stress tensor has been developed by Eringen and Maugin (1990a; 1990b), while the second order theory has been developed by Ko and Dulikravich (1998). If we limit the analysis to the linear fluid medium, then we can use the following expressions.

$$\underline{P} = \epsilon_p \hat{\underline{E}}, \quad (13)$$

$$\hat{\underline{M}} = \frac{1}{\mu_m} \underline{B}, \quad (14)$$

$$\underline{J}_c = \sigma \hat{\underline{E}} + \sigma_T \nabla T, \quad (15)$$

$$\underline{q} = \kappa \nabla T + \kappa_E \hat{\underline{E}}, \quad (16)$$

$$\underline{\underline{t}} = -\tilde{p} \underline{I} + \eta [\nabla \underline{v} + (\nabla \underline{v})^t]. \quad (17)$$

Here, all the coefficients are material properties and depend only on temperature, T, because the fluid is assumed to be incompressible. The extension of the analysis to the non-linear fluids makes the formulation very complicated and cumbersome (Dulikravich, 1998; Ko and Dulikravich, 1998).

Substituting the constitutive relations into the electromagnetic (Eqs. (3)-(7)) and thermo-mechanical balance laws (Eqs. (10) - (12)), the following closed set of governing equations can be obtained

$$\begin{aligned} \epsilon_0 \frac{\partial \underline{E}}{\partial t} = & \nabla \times \left(\frac{1}{\mu} \underline{B} + \epsilon_p \underline{v} \times (\underline{E} + \underline{v} \times \underline{B}) \right) - q_e \underline{v} \\ & - (\sigma (\underline{E} + \underline{v} \times \underline{B}) + \sigma_T \nabla T) - \frac{\partial \underline{P}}{\partial t} \equiv \underline{F}_D - \frac{\partial \underline{P}}{\partial t} \end{aligned} \quad (18)$$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} \quad (19)$$

$$\frac{\partial q_e}{\partial t} = -\nabla \cdot (q_e \underline{v} + \sigma (\underline{E} + \underline{v} \times \underline{B}) + \sigma_T \nabla T) \quad (20)$$

$$\nabla \cdot \underline{v} = 0 \quad (21)$$

$$\begin{aligned} \rho \frac{\partial \underline{v}}{\partial t} = & -\rho (\underline{v} \cdot \nabla) \underline{v} - \nabla \tilde{p} + \nabla \cdot (\eta (\nabla \underline{v} + (\nabla \underline{v})^t)) + \rho \underline{f} + q_e \underline{E} \\ & + (q_e \underline{v} + \sigma (\underline{E} + \underline{v} \times \underline{B}) + \sigma_T \nabla T) \times \underline{B} + \epsilon_p (\nabla \underline{E}) \cdot (\underline{E} + \underline{v} \times \underline{B}) \\ & + \nabla \cdot (\epsilon_p \underline{v} (\underline{E} + \underline{v} \times \underline{B})) + (\nabla \underline{B}) \cdot \left(\frac{\underline{B}}{\mu_m} - \epsilon_p \underline{v} \times (\underline{E} + \underline{v} \times \underline{B}) \right) \\ & - \epsilon_p (\underline{E} + \underline{v} \times \underline{B}) \times (\nabla \times \underline{E}) + \frac{\partial \underline{P}}{\partial t} \times \underline{B} \equiv \rho \underline{F}_v + \frac{\partial \underline{P}}{\partial t} \times \underline{B} \end{aligned} \quad (22)$$

$$\begin{aligned} \rho C_p \frac{\partial T}{\partial t} = & -\rho C_p (\underline{v} \cdot \nabla) T + \nabla \cdot (\kappa \nabla T + \kappa_E (\underline{E} + \underline{v} \times \underline{B})) + Q_h \\ & + (\underline{E} + \underline{v} \times \underline{B}) \cdot (\sigma (\underline{E} + \underline{v} \times \underline{B}) + \sigma_T \nabla T) + \frac{\underline{B}}{\mu_m} \cdot (\nabla \times \underline{E}) \\ & - \frac{\underline{B}}{\mu_m} \cdot ((\underline{v} \cdot \nabla) \underline{B}) + (\underline{E} + \underline{v} \times \underline{B}) \cdot ((\underline{v} \cdot \nabla) (\epsilon_p (\underline{E} + \underline{v} \times \underline{B}))) \\ & + (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial \underline{P}}{\partial t} \equiv \rho C_p F_T + (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial \underline{P}}{\partial t} \end{aligned} \quad (23)$$

$$\frac{\partial \underline{P}}{\partial t} = \epsilon_p \left(\frac{\partial \underline{E}}{\partial t} + \frac{\partial \underline{v}}{\partial t} \times \underline{B} - \underline{v} \times (\nabla \times \underline{E}) \right) + \epsilon' \frac{\partial T}{\partial t} (\underline{E} + \underline{v} \times \underline{B}) \quad (24)$$

Here, note that $\frac{d\epsilon_p}{dT} = \frac{d\epsilon}{dT} \equiv \epsilon'$. This set of equations is not in a standard form since certain time derivative terms exist on the right hand side (Eqs. (18), (22)-(24)). Therefore, we must solve for $\frac{\partial \underline{E}}{\partial t}$, $\frac{\partial \underline{v}}{\partial t}$, $\frac{\partial T}{\partial t}$ and $\frac{\partial \underline{P}}{\partial t}$ in order to obtain a system of equations that could be solved by marching in time. After a lengthy derivation, the following system of evolution equations can be formulated.

$$\frac{\partial \underline{E}}{\partial t} = \frac{\underline{F}_D}{\epsilon_0} - \frac{1}{\epsilon_0 G_0} (\underline{F}_0 + G_1 \underline{B} + \epsilon' G_2 \hat{\underline{E}}) \quad (25)$$

$$\frac{\partial \underline{v}}{\partial t} = \underline{F}_v - \frac{1}{\rho G_0} (\underline{B} \times \underline{F}_0 + \epsilon' G_2 \underline{B} \times \hat{\underline{E}}) \quad (26)$$

$$\frac{\partial T}{\partial t} = G_2 \quad (27)$$

$$\frac{\partial \underline{P}}{\partial t} = \frac{1}{G_0} (\underline{F}_0 + G_1 \underline{B} + \epsilon' G_2 \hat{\underline{E}}) \quad (28)$$

$$\underline{F}_0 = (\epsilon_r - 1) \underline{E}_D - \epsilon_p (\underline{B} \times \underline{F}_v + \underline{v} \times (\nabla \times \underline{E})), \quad (29)$$

$$G_0 = \epsilon_r + \frac{\epsilon_p}{\rho} (\underline{\mathbf{B}} \cdot \underline{\mathbf{B}}), \quad G_1 = \frac{\epsilon_p \Delta_1}{\rho \Delta_0}, \quad G_2 = F_T + \frac{\Delta_2}{\rho C_p \Delta_0}, \quad (30)$$

$$\Delta_0 = \epsilon_r G_0 - \frac{\epsilon'}{\rho C_p} (\epsilon_r (\hat{\underline{\mathbf{E}}} \cdot \hat{\underline{\mathbf{E}}}) + \frac{\epsilon_p}{\rho} (\underline{\mathbf{B}} \cdot \hat{\underline{\mathbf{E}}})^2), \quad (31)$$

$$\Delta_1 = G_0 (\underline{\mathbf{B}} \cdot \underline{\mathbf{F}}_0) + \frac{\epsilon'}{\rho C_p} (\rho C_p G_0 F_T (\underline{\mathbf{B}} \cdot \hat{\underline{\mathbf{E}}}) - (\hat{\underline{\mathbf{E}}} \cdot \hat{\underline{\mathbf{E}}}) (\underline{\mathbf{B}} \cdot \underline{\mathbf{F}}_0) + (\underline{\mathbf{B}} \cdot \hat{\underline{\mathbf{E}}}) (\hat{\underline{\mathbf{E}}} \cdot \underline{\mathbf{F}}_0)) \quad (32)$$

$$\Delta_2 = \epsilon_r (\hat{\underline{\mathbf{E}}} \cdot \underline{\mathbf{F}}_0) + \frac{\epsilon_p}{\rho} (\underline{\mathbf{B}} \cdot \hat{\underline{\mathbf{E}}}) (\underline{\mathbf{B}} \cdot \underline{\mathbf{F}}_0) + \epsilon' F_T (\epsilon_r (\hat{\underline{\mathbf{E}}} \cdot \hat{\underline{\mathbf{E}}}) + \frac{\epsilon_p}{\rho} (\underline{\mathbf{B}} \cdot \hat{\underline{\mathbf{E}}})^2). \quad (33)$$

TWO-DIMENSIONAL PLANAR EMHD FLOWS

The evolution equations for the general cases of EMHD flows are too complicated to be used for practical applications. Here, we will deal with a simpler case, that is, a two-dimensional case where the flow is confined in the x-y plane. The velocity and electric field (influencing $\underline{\mathbf{E}}, \hat{\underline{\mathbf{E}}}, \underline{\mathbf{P}}, \underline{\mathbf{J}}, \underline{\mathbf{J}}$) have only the x and y components, while the magnetic field (influencing $\underline{\mathbf{B}}, \underline{\mathbf{H}}, \underline{\mathbf{M}}, \hat{\underline{\mathbf{M}}}$) has only the z component. None of the variables depends on z-coordinate. More specifically,

$$\underline{\mathbf{E}} = \{E_1, E_2, 0\}^t, \quad \underline{\mathbf{B}} = \{0, 0, B_3\}^t, \quad \underline{\mathbf{v}} = \{V_1, V_2, 0\}^t. \quad (34)$$

Notice that the conservation of the magnetic field (Eq. (2)) is automatically satisfied in this case. In addition to the assumption of two-dimensional flow, we assume that the variation of material properties such as electric permittivity, magnetic permeability, and viscosity are negligible.

Substituting the reduced form of Eq. (34) into Eqs. (18)-(23) and utilizing the general solution given in Eqs. (25)-(28), a system of accurate evolution equations corresponding to the two-dimensional EMHD flow with constant material properties can be obtained. The resulting system, however, is extremely complicated and severely coupled. It is nearly impossible to get a physical meaning about the characteristic behavior of such EMHD flows, because the system does not allow further analytical treatment.

If the analysis is limited to the formulation of non-reflective boundary conditions needed in simulating the steady state flows, then this difficulty can be avoided by slightly modifying the governing equations. For the simulation of steady state EMHD flow, the time derivative of the polarization vector in the right hand sides of Eqs. (18), (22), (23) can be omitted because it vanishes at the converged final steady state. This idea is similar to the preconditioning method that is frequently used to accelerate the convergence of iterative algorithms. On the other hand, we can add the so-called artificial compressibility term in the continuity equation in order to create an artificially unsteady term for the marching in time as follows (Chorin, 1967):

$$\frac{\partial \bar{p}}{\partial t} + \beta \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} \right) = 0. \quad (35)$$

Keeping in mind the form of Eq. (34), the modified version of the remaining equations can be readily obtained by omitting the time derivative of the polarization vector. The specific form of the resulting system will be presented in the next section.

NON-REFLECTIVE BOUNDARY CONDITIONS

It is ideal to derive the non-reflective boundary conditions from the characteristic form of the full EMHD system, because this can provide the maximum information about the interactions of the electromagnetic and thermo-mechanical effects. Unfortunately, the full system, even for two-dimensional case, is too involved to treat it in a coupled manner. The coefficient matrix in the characteristic form of the full governing equation system is so dense that it does not allow the symbolic (closed form) calculation of eigenvalues. In addition to this complication, the fact that electromagnetic time scale differs from the thermo-mechanical one by orders of magnitude makes the coupled analysis approach difficult. For these reasons, the formulation of the characteristic boundary conditions will be carried out separately for the two subsystems (electromechanical and thermo-mechanical).

Non-reflective Boundary Conditions for Electromagnetic Subsystem

The characteristic form of the electromagnetic (Maxwell's) subsystem can be written from Eqs. (18)-(20) as follows.

$$\frac{\partial \underline{\mathbf{Q}}^1}{\partial t} + \underline{\mathbf{A}}^{11} \frac{\partial \underline{\mathbf{Q}}^1}{\partial x} + \underline{\mathbf{B}}^{11} \frac{\partial \underline{\mathbf{Q}}^1}{\partial y} = \underline{\mathbf{S}}^1 - \underline{\mathbf{A}}^{12} \frac{\partial \underline{\mathbf{Q}}^2}{\partial x} - \underline{\mathbf{B}}^{12} \frac{\partial \underline{\mathbf{Q}}^2}{\partial y} \equiv \tilde{\underline{\mathbf{S}}}^1 \quad (36)$$

where indices 1 and 2 correspond to the electromagnetic and thermo-mechanical subsystem, respectively. The electromagnetic solution vector represents the unknown primitive variables

$$\underline{\mathbf{Q}}^1 = \{E_1, E_2, B_3, q_e\}^t, \quad (37)$$

while the coefficient matrix $\underline{\mathbf{A}}^{11}$ is defined by

$$\underline{\mathbf{A}}^{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -(\epsilon_r - 1)V_2 & (\epsilon_r - 1)V_1 & 1/\epsilon_0 \mu & 0 \\ 0 & 1 & 0 & 0 \\ \sigma & 0 & \sigma V_2 & V_1 \end{bmatrix}. \quad (38)$$

The remaining coefficient matrices $\underline{\mathbf{B}}^{11}$, $\underline{\mathbf{A}}^{12}$, and $\underline{\mathbf{B}}^{12}$ may be obtained by collecting the appropriate first order spatial derivative terms in Eqs. (18)-(20). The details will be omitted because of the restricted typing space. The source vector, $\underline{\mathbf{S}}^1$, is defined as

$$\underline{\mathbf{S}}^1 = \{-(q_e V_1 + \sigma \hat{\underline{\mathbf{E}}}_1)/\epsilon_0, -(q_e V_2 + \sigma \hat{\underline{\mathbf{E}}}_2)/\epsilon_0, 0, -\sigma_T \nabla^2 T\}^t. \quad (39)$$

Because the source vector should not contain any spatial derivative, neither \underline{S}^1 nor $\tilde{\underline{S}}^1$ is a genuine source vector. A second order derivative (heat diffusion) term is included in them. This means that we neglect the diffusive nature of the characteristic wave. This is the only viable option since there is no known method to handle this effect.

The eigenmatrix \underline{A}^{11} corresponding to the coefficient matrix \underline{A}^{11} can be calculated as

$$\underline{A}^{11} = \text{diag}\{\lambda_1^{11}, \lambda_2^{11}, \lambda_3^{11}, \lambda_4^{11}\} = \text{diag}\{0, V_1, \xi_1, \xi_2\}, \quad (40)$$

$$\xi_{1,2} = \frac{\sqrt{\mu\epsilon_p} V_1 \pm \sqrt{4\epsilon_0 + \mu\epsilon_p^2} V_1^2}{2\epsilon_0\sqrt{\mu}}. \quad (41)$$

These eigenvalues show that the incoming and outgoing electromagnetic waves are not influenced by the electromagnetic field intensity, but by the fluid motion and the fluid material properties.

The matrix of left eigenvectors corresponding to \underline{A}^{11} , which is identical to the inverse of the similarity transformation matrix, can be calculated as

$$\underline{N}^{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\sigma(1+b_1+\epsilon_p\mu V_2^2)}{V_1(1+b_1)} & \frac{-\sigma\epsilon_0\mu V_2}{1+b_1} & \frac{\sigma b_1}{V_2(1+b_1)} & 1 \\ -\epsilon_p\mu V_2 & \epsilon_0\mu\xi_1 & 1 & 0 \\ -\epsilon_p\mu V_2 & \epsilon_0\mu\xi_2 & 1 & 0 \end{bmatrix}, \quad (42)$$

$$b_1 = \mu(\epsilon_p - \epsilon_0)V_1^2. \quad (43)$$

The direction of wave propagation is well defined for the actually or locally one-dimensional problems. For two-dimensional problems, there is no unique direction of propagation. This is reflected in the fact that \underline{A}^{11} and \underline{B}^{11} cannot be simultaneously diagonalized. For most cases, however, there exists a main flow direction at the inlet and outlet boundaries where the characteristic boundary conditions are required. If x-direction is such a direction, then one can estimate that the wave propagation characteristics are predominantly determined by \underline{A}^{11} , and the effect of the transversal variation is relatively negligible.

In this (quasi one-dimensional) approach, the transverse term is considered constant and treated as a source term in the same way as the thermo-mechanical derivative terms are treated as source terms (Thompson, 1987; Dulikravich and Jing, 1997).

The fact that all eigenvalues of \underline{A}^{11} are real means that Eq. (36) is locally hyperbolic. Following Thompson's approach, the boundary conditions at the inlet and outlet can be written as

$$\underline{n}_i \frac{\partial \underline{Q}^1}{\partial t} + \delta_i \lambda_i^{11} \underline{n}_i \frac{\partial \underline{Q}^1}{\partial x} + \underline{n}_i \underline{U}^1 = 0 \text{ at } x = x_{in}, x_{out} \quad (44)$$

$$\delta_i = \begin{cases} 1 & \text{for outgoing waves,} \\ 0 & \text{for incoming waves.} \end{cases} \quad (45)$$

Here, \underline{n}_i is the i-th row vector of \underline{N}^{11} , and the source term modification vector \underline{U}^1 is given by

$$\underline{U}^1 = \underline{B}^{11} \frac{\partial \underline{Q}^1}{\partial y} - \tilde{\underline{S}}^1. \quad (46)$$

Non-reflective Boundary Conditions for Thermo-mechanical Subsystem

The procedure of formulating the non-reflective boundary conditions for the thermo-mechanical subsystem is the same as that for the electromagnetic subsystem. Using Eqs. (35), (22), and (23), this subsystem can be written in a characteristic form as

$$\frac{\partial \underline{Q}^2}{\partial t} + \underline{A}^{22} \frac{\partial \underline{Q}^2}{\partial x} + \underline{B}^{22} \frac{\partial \underline{Q}^2}{\partial y} = \underline{S}^2 - \underline{A}^{21} \frac{\partial \underline{Q}^1}{\partial x} - \underline{B}^{21} \frac{\partial \underline{Q}^1}{\partial y} \equiv \tilde{\underline{S}}^2, \quad (47)$$

where the solution vector and source vector are defined by

$$\underline{Q}^2 = \{\bar{p}, V_1, V_2, T\}^t, \quad (48)$$

$$\begin{aligned} \underline{S}^2 = \{ & 0, (\eta \nabla^2 V_1 + \rho f_1 + q_e \hat{E}_1 + \sigma \hat{E}_2 B_3) / \rho, \\ & (\eta \nabla^2 V_2 + \rho f_2 + q_e \hat{E}_2 - \sigma \hat{E}_1 B_3) / \rho, \\ & (\kappa \nabla^2 T + Q_h + \sigma(\hat{E}_1^2 + \hat{E}_2^2) / \rho C_p) \}^t \end{aligned} \quad (49)$$

It should be pointed out that the source vector has linear momentum and heat diffusion terms. Suppose that the dominant wave propagation is in the x-direction. For such cases, prevailing role is played by the coefficient matrix \underline{A}^{22} that is given as

$$\underline{A}^{22} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1/\rho & G V_1 & 0 & 0 \\ 0 & 0 & G V_1 & \sigma_T B_3 / \rho \\ 0 & \frac{\epsilon_p \hat{E}_2 V_1 B_3}{\rho C_p} & \frac{-\epsilon_p \hat{E}_1 V_1 B_3 - \kappa_E B_3}{\rho C_p} & V_1 - \frac{\sigma_T \hat{E}_1}{\rho C_p} \end{bmatrix} \quad (50)$$

$$G = 1 + \frac{\epsilon_p B_3^2}{\rho}, \quad C = \rho C_p. \quad (51)$$

The eigenmatrix \underline{A}^{22} corresponding to \underline{A}^{22} can be calculated as

$$\underline{A}^{22} = \text{diag}\{\lambda_1^{22}, \lambda_2^{22}, \lambda_3^{22}, \lambda_4^{22}\}, \quad (52)$$

$$\lambda_{1,2}^{22} = \frac{G V_1}{2} \pm \sqrt{\frac{\beta}{\rho} + \left(\frac{G V_1}{2}\right)^2} \quad (53)$$

$$\lambda_{3,4}^{22} = \frac{C(1+G)V_1 - \sigma_T \hat{E}_1}{2C} \pm \frac{\sqrt{(C(G-1)V_1 + \sigma_T \hat{E}_1)^2 - 4C(\kappa_E + \varepsilon_p V_1 \hat{E}_1)B_3^2 \sigma_T / \rho}}{2C} \quad (54)$$

The thermo-mechanical subsystem eigenvalues are strongly dependent on the electric and magnetic field intensities as well as the velocity and fluid material property. This characteristic is rather different from the electromagnetic subsystem. It implies that the effect of the electromagnetic field on flow-field is much stronger than that of the flow-field on the electromagnetic field. The most important factor combining the electromagnetic and thermo-mechanical effects is the thermo-electric conductivity, σ_T . The analysis leading to the boundary conditions for the thermo-mechanical subsystem could be performed in the same fashion as shown in Eqs. (42)-(46). Since the result is more complicated than in the case of the electromagnetic subsystem (compare the eigenmatrices), we omit the details.

CONCLUSIONS

Starting from a consistent EMHD model with linear constitutive equations, a standard form of governing equation system was obtained in Cartesian coordinates. Introducing the artificial compressibility and neglecting the temporal variation of polarization vector, both Maxwell's equations (electromagnetic subsystem) and modified Navier-Stokes equations (thermo-mechanical subsystem) were transformed to their characteristic forms for the fluids with constant property. The characteristic and non-reflective boundary conditions at the inlet and outlet boundary were formulated for both subsystems by assuming that transversal variation is negligible. The results for the eigenvalues of the coefficient matrices show that there is a substantial coupling between the electromagnetic field and flow-field. Since the time derivative of the polarization was neglected and artificial compressibility was introduced, the results are applicable only to the steady state EMHD planar flows. For the simulation of unsteady flows, further studies are needed to circumvent this restriction.

ACKNOWLEDGMENTS

The authors are grateful for the partial support by the ALCOA Foundation Grant facilitated by Dr. Yimin Ruan and Dr. Owen Richmond of ALCOA Technical Center, and for the partial support by a National Science Foundation grant DMI-9700040 monitored by Dr. George A. Hazelrigg and a Lawrence Livermore National Laboratory grant LLNL-B344847 monitored by Dr. J. Ray Smith. Professor Akhlesh Lakhtakia of The Pennsylvania State University provided valuable suggestions in the final phases of this work.

REFERENCES

- Chorin, A.J., 1967, "A Numerical Method of Solving Incompressible Viscous Flow Problems," *Journal of Computational Physics*, Vol. 2, pp. 12-26.
- Dulikravich, G.S., 1998, "Electro-Magneto-Hydrodynamics and Solidification", Chapter 9 in *Advances in the Flow and Rheology of Non-Newtonian Fluids*, Editors: Siginer, D.A., DeKee, D. and Chhabra, R.P., Elsevier Science, The Netherlands.
- Dulikravich, G.S., and Jing, Y.-H., 1996, "Fully Conservative Forms of a System of Unified Electro-Magneto-Hydrodynamic

Equations", *Symposium on Rheology and Fluid Mechanics of Nonlinear Materials*, Editors: D.A. Siginer and S.G. Advani, ASME IMECE, Atlanta, GA, Nov. 17-22, 1996, AMD-Vol.217, pp.309-314.

Dulikravich, G.S., and Jing, Y.-H., 1997, "Boundary Conditions for Electro-Magneto-Hydrodynamics", *Symposium on Rheology and Fluid Mechanics of Non-Linear Materials*, Editors: S.G. Advani and D.A. Siginer, ASME IMECE, Dallas, TX, Nov. 16-21, 1997, ASME FED-Vol. 243/MD-Vol. 78, pp. 101-117.

Dulikravich, G.S., and Lynn, S.R., 1995a, "Electro-Magneto-Hydrodynamics: Part I - Introductory Concepts", *Symposium on Development in Electrorheological Flows*, ASME IMECE, San Francisco, CA, Nov. 1995, Editors: D.A. Siginer and G.S. Dulikravich, ASME FED-Vol. 235, MD-Vol. 71, pp. 49-58.

Dulikravich, G.S., and Lynn, S.R., 1995b, "Electro-Magneto-Hydrodynamics: Part II - A Survey of Mathematical Models", *Symposium on Development in Electrorheological Flows*, ASME IMECE, San Francisco, CA, Nov. 1995, Editors: D.A. Siginer and G.S. Dulikravich, ASME FED-Vol. 235, MD-Vol. 71, pp. 59-70.

Dulikravich, G.S., and Lynn, S.R., 1997a, "Unified Electro-Magneto-Fluid Dynamics (EMFD): Introductory Concepts", *International Journal of Non-linear Mechanics*, Vol. 32, No.5, Sept. 1997, pp. 913-922.

Dulikravich, G.S., and Lynn, S.R., 1997b, "Unified Electro-Magneto-Fluid Dynamics (EMFD): Mathematical Models", *International Journal of Non-linear Mechanics*, Vol. 32, No.5, Sept. 1997, pp. 923-932.

Eringen, A.C., and Maugin, G.A., 1990a, *Electrodynamics of Continua I; Foundations and Solid Media*, Springer-Verlag, New York.

Eringen, A.C., and Maugin, G.A., 1990b, *Electrodynamics of Continua II; Fluids and Complex Media*, Springer-Verlag, New York.

Ko, H.-J., and Dulikravich, G.S., 1998, "A Fully Non-linear Theory of Electro-Magneto-Hydrodynamics", *Symposium on Rheology and Fluid Mechanics of Nonlinear Materials*, ASME, Anaheim, CA, Nov. 1998, Editors: D.A. Siginer and D.D. Kee, ASME FED-Vol. 246, MD-Vol. 81, pp. 173-182.

Lakhtakia, A., 1993, "Frequency-Dependent Continuum Electromagnetic Properties of a Gas of Scattering Centers," *Advances in Chemical Physics*, Vol. 85, pt. 2, pp. 311-359.

Melcher, J.R., 1981, *Continuum Electromechanics*, MIT Press, Cambridge, MA.

Reitsma, S.H., Manno, V.P., and Tureaud, T.F., 1993, "Non-reflective Boundary Conditions for Fluid-Acoustic Simulation Using a Finite Volume Formulation", *Symposium on Computational Aero- and Hydro-Acoustics*, ASME FED-Vol. 147, pp. 71-82.

Shang, J.S., 1991, "Characteristic- Based Methods for the Time-Domain Maxwell Equations," AIAA paper, 91-0606, Reno, NV.

Shankar, V.J., Hall, W.F. and Alireza, H.M., 1989, "A Time Domain Differential Solver for Electromagnetic Scattering," *Proceedings of IEEE*, Vol. 77, no. 5, pp. 709-721.

Sun, M.T., Wu, S.T., and Dryer, M., 1995, "On the Time-Dependent Numerical Boundary Conditions of Magnetohydrodynamic Flows," *Journal of Computational Physics*, Vol. 116, pp. 330-342.

Thompson, K.W., 1987, "Time Dependent Boundary Conditions for Hyperbolic System, I," *J. of Computational Physics*, Vol. 68, pp. 1-24.

Thompson, K.W., 1990, "Time Dependent Boundary Conditions for Hyperbolic System, II," *Journal of Computational Physics*, Vol. 89, pp. 439-461.

Wineman, A.S., and Rajagopal, K.R., 1995, "On Constitutive Equations for Electrorheological Materials", *Continuum Mechanics and Thermodynamics*, Springer-Verlag, Vol. 7, pp. 1-23.