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MULTIDISCIPLINARY INVERSE PROBLEMS

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ABSTRACT

This paper presents a limited survey of methods and multidisciplinary applications of various techniques for the solution of several classes of inverse problems as developed and practiced by our research team. Sketches of solution methods for inverse problems of shape determination, boundary conditions determination, sources determination, and physical properties determination are presented from the fields of aerodynamics, heat transfer, elasticity, and electrostatics.

GENERAL INTRODUCTION

Engineering field problems are defined (Kubo, 1993) by the governing partial differential or integral equation(s), shape(s) and size(s) of the domain(s), boundary and initial conditions, material properties of the media contained in the field, and by internal sources and external forces or inputs. If all of this information is known, the field problem is of an analysis (or direct) type and generally considered as well posed and solvable. If any of this information is unknown or unavailable, the field problem becomes an indirect (or inverse) problem and is generally considered to be ill posed and unsolvable. Specifically, inverse problems can be classified as:

1. *Shape determination inverse problems,*
2. *Boundary/initial value determination inverse problems,*
3. *Sources and forces determination inverse problems,*
4. *Material properties determination inverse problems, and*
5. *Governing equation(s) determination inverse problems.*

The inverse problems are solvable if additional information is provided and if appropriate numerical algorithms are used. The objective of this paper is to offer a very brief survey of research on the solution methods for multidisciplinary inverse problems that has been performed in our Multidisciplinary Analysis, Inverse Design and Optimization (MAIDO) Laboratory.

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1. SHAPE DETERMINATION INVERSE PROBLEMS

The problem of determining sizes, shapes, and locations of objects or cavities inside a given object sounds like a formidable task. In reality, this type of inverse problem is probably the most common. The problem can be solved only if certain field quantity (pressure, heat flux, stress, magnetic field, etc.) can be specified on these unknown boundaries in addition to their complementary field quantities (velocity, temperature, deformation, electric field, etc.) on the same boundaries.

1.1 Aerodynamic Shape Inverse Design

A typical inverse aerodynamic shape design is defined as follows: if a desirable fluid pressure distribution is specified on the yet unknown surface of an aerodynamic body, find the shape of the body that will produce this pressure distribution subject to the specified global flow-field conditions. Two classes of tools for inverse aerodynamic shape design are: a) methods with coupled shape modification and flow-field analysis, and b) methods with uncoupled shape modification and flow-field analysis (Dulikravich, 1984; 1987; 1991; 1992; 1995; 1997; Fujii and Dulikravich, 1999; Tanaka and Dulikravich, 1998).

The coupled inverse shape design methods require special consideration in the writing of the flow-field analysis computer code. These software modifications represent a major undertaking, even if the source version of the flow-field analysis code is available. For example, an indirect surface transpiration technique might need to exchange no-slip wall boundary conditions on the body surface with specified pressure boundary conditions in order to obtain a shape update. Other examples of this class of design techniques are: stream-function-as-coordinate formulation, characteristic boundary condition concept, integro-differential equation concept, fictitious gas concept, direct surface transpiration concept, and adjoint operator/control theory approaches. Furthermore, most of the existing inverse shape design methods are not applicable to viscous flows or to three-dimensional configurations.

The uncoupled inverse shape design methods require no modification to an existing flow-field analysis computer code. This means that any flow-field analysis code (a panel code, an Euler code, a Navier-Stokes code, or even surface pressures obtained experimentally from a wind tunnel testing) can be used in the uncoupled aerodynamic shape inverse design without a need for alterations of such an analysis tool. For example, elastic membrane technique and DISC method (Dulikravich, 1995; 1997) require knowledge of only the surface pressure distribution on the current aerodynamic shape as an output from the flow-field analysis code in order to predict a shape update.

It should be pointed out that inverse methods for aerodynamic shape design are capable of creating only point-designs, that is, the resulting shapes will have the desired aerodynamic characteristics only at the design conditions. If the angle of attack, free stream Mach number, etc. in actual flight situations is different from the values used in the design, the aerodynamic performance will deteriorate sometimes quite dramatically especially at transonic speeds.

1.2 Elastic Surface Motion Concept

This technique (Dulikravich, 1995; 1997) treats the surface of an aerodynamic body as an elastic membrane that deforms under aerodynamic loads until it achieves a desired (target) distribution of surface pressure coefficient, C_p. The original non-physical model for the evolution of a two-dimensional shape was given by

$$\beta_0 \Delta n + \beta_1 \frac{d\Delta n}{dx} + \beta_2 \frac{d^2 \Delta n}{dx^2} = C_p^{target} - C_p^{actual} \qquad (1)$$

Here, Δn's are defined as shape corrections along outward normal vectors to the airfoil contour. Coefficients β_0, β_1, and β_2 are user supplied constants that control the rate of convergence of the airfoil shape. This technique was modified by Malone et al. (1987), giving

$$\beta_0 \Delta y + \beta_1 \frac{d\Delta y}{dx} + \beta_2 \frac{d^2 \Delta y}{dx^2} = C_p^{target} - C_p^{actual} \qquad (2)$$

With this formulation (dubbed MGM for modified-Garabedian-McFadden or Malone-Garabedian-McFadden) all shape modifications are in the y-direction, thus preventing the chord length from changing in the x-direction. The MGM shape evolution equation (2) is traditionally solved for shape corrections, Δy, by evaluating the derivatives in Eq. (2) using a finite differencing. Two major problems with the classical MGM approach are its slow convergence at the leading and trailing edges of the airfoil, and its significantly slower convergence in conjunction with the flow-field analysis codes of increasing non-linearity (Dulikravich and Baker, 1999a).

It has been observed in practice, and can be shown analytically that the radius of convergence of the iterative matrix in the present formulation of this method depends on the non-linearity of the flow-field analysis module. This means that when using a progressively more non-linear flow-field analysis, this inverse shape design method will unavoidably need between two and three orders of magnitude more flow-field analysis runs than when using a simple linear panel code. For example, a two-dimensional airfoil shape inverse design with this method utilizing a Navier-Stokes flow analysis code may require over ten thousand calls to the Navier-Stokes code. This is obviously unacceptable for three-dimensional applications.

1.3 Fourier Series Elastic Membrane Formulation

The slow convergence problem of the classical MGM technique can be eliminated with a new formulation of the elastic membrane design concept that allows a Fourier series analytical solution to the shape evolution equation (Dulikravich and Baker, 1999a; 1999b). Notice that Eq. (2) can be expressed on the upper airfoil contour as

$$-\beta_0 \Delta y + \beta_s \frac{d\Delta y}{ds} + \beta_{ss} \frac{d^2 \Delta y}{ds^2} = \Delta C_p \, , \qquad (3)$$

where s is the airfoil contour-following coordinate, and as

$$\beta_0 \Delta y + \beta_s \frac{d\Delta y}{ds} - \beta_{ss} \frac{d^2 \Delta y}{ds^2} = \Delta C_p \qquad (4)$$

on the lower airfoil contour.

These two ordinary differential equations with constant coefficients are of a simple linear forced mass-damper-spring system type where the monotonically increasing time coordinate in the forced mass-damper-spring system and the monotonically increasing contour following coordinate, s, in Eqs. (3) and (4) are equivalents. There is also an analogy between the forcing function in the mass-damper-spring system, which varies arbitrarily with time and the surface pressure coefficient difference, ΔC_p, which varies arbitrarily with the contour following coordinate, s, in Eqs. (3) and (4). Notice also a global periodicity of the mass-damper-spring forcing function and the surface pressure coefficient difference, ΔC_p, distribution that repeats its value at the starting and the ending contour-following s-coordinate location (typically the trailing edge point). The arbitrary surface distribution of ΔC_p in Eqs. (3) and (4) can be represented by utilizing the Fourier series expansion as

$$\Delta C_p(s) = a_0 + \sum_{n=1}^{n_{max}} \left[a_n \cos(N_n s) + b_n \sin(N_n s) \right] \qquad (5)$$

where

$$N_n = \frac{2n\pi}{L} \tag{6}$$

and L is the total length of the airfoil contour. The particular solution of either Eq. (3) or Eq. (4) can be represented in the general Fourier series form as

$$\Delta y_p = A_0 + \sum_{n=1}^{n_{max}} [A_n \cos(N_n s) + B_n \sin(N_n s)] \tag{7}$$

Substitution of Eqs. (5)-(7) and analytical derivatives of Eq. (7) into the airfoil top contour evolution equation (3) and bottom contour evolution equation (4) yields

$$A_n^{top} = \frac{a_n(\beta_0 + N_n{}^2\beta_{ss}) - b_n(\beta_s N_n)}{(\beta_0 + N_n^2\beta_{ss})^2 + (\beta_s N_n)^2}, n = 0,1,2,\cdots \tag{8}$$

$$B_n^{top} = \frac{b_n(\beta_0 + N_n^2\beta_{ss}) + a_n(\beta_s N_n)}{(\beta_0 + N_n^2\beta_{ss})^2 + (\beta_s N_n)^2}, n = 1,2,3,\cdots \tag{9}$$

$$A_n^{bottom} = \frac{a_n(-\beta_0 - N_n{}^2\beta_{ss}) - b_n(\beta_s N_n)}{(\beta_0 + N_n{}^2\beta_{ss})^2 + (\beta_s N_n)^2}, n = 0,1,2,\cdots \tag{10}$$

$$B_n^{bottom} = \frac{b_n(-\beta_0 - N_n{}^2\beta_{ss}) + a_n(\beta_s N_n)}{(\beta_0 + N_n{}^2\beta_{ss})^2 + (\beta_s N_n)^2}, n = 1,2,3,\cdots \tag{11}$$

Since the Fourier coefficients of the particular solutions on the upper and lower airfoil contours are different, it can be expected that gaps will form at the leading and trailing edges of the airfoil. These gaps can be closed with appropriate homogeneous solutions to Eqs. (3) and (4). An analytical form of the homogenous solution for the airfoil upper contour is

$$\Delta y_h^{top} = F^{top} e^{\lambda_1 s} + G^{top} e^{\lambda_2 s} \tag{12}$$

with a similar expression for the airfoil lower contour. Here,

$$\lambda_{1,2} = \frac{\beta_s \pm \sqrt{\beta_s{}^2 + 4\beta_0\beta_{ss}}}{2\beta_{ss}} \tag{13}$$

Thus, the overall displacement (correction) of the airfoil contour is given by the following equations:

$$\Delta y^{top} = F^{top} e^{\lambda_1 s} + G^{top} e^{\lambda_2 s} + \sum_{n=0}^{n_{max}} \left[A_n^{top} \cos(N_n s) + B_n^{top} \sin(N_n s)\right] \tag{14}$$

$$\Delta y^{bottom} = F^{bottom} e^{-\lambda_1 s} + G^{bottom} e^{-\lambda_2 s} + \sum_{n=0}^{n_{max}} \left[A_n^{bottom} \cos(N_n s) + B_n^{bottom} \sin(N_n s)\right] \tag{15}$$

The four unknown constants F and G can now be determined for the upper and lower airfoil contours such that the following four boundary conditions: zero trailing edge displacement, trailing edge closure, leading edge closure, and smooth leading edge deformation. Simultaneous solution of these four conditions for the unknown coefficients F and G results in

$$\begin{Bmatrix} F^{bottom} \\ G^{bottom} \\ F^{top} \\ G^{top} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{L\lambda_1} & e^{L\lambda_2} \\ e^{-s_{LE}\lambda_1} & e^{-s_{LE}\lambda_2} & -e^{s_{LE}\lambda_1} & -e^{s_{LE}\lambda_2} \\ -\lambda_1 e^{-s_{LE}\lambda_1} & -\lambda_2 e^{-s_{LE}\lambda_2} & -\lambda_1 e^{s_{LE}\lambda_1} & -\lambda_2 e^{s_{LE}\lambda_2} \end{bmatrix}^{-1} \cdot \begin{Bmatrix} -\sum_{n=0}^{n_{max}} A_n^{bottom} \\ -\sum_{n=0}^{n_{max}} A_n^{top} \\ \sum_{n=0}^{n_{max}} [\Delta A_n \cos(N_n s_{LE}) + \Delta B_n \sin(N_n s_{LE})] \\ \sum_{n=0}^{n_{max}} [-N_n \Delta A_n \sin(N_n s_{LE}) + N_n \Delta B_n \cos(N_n s_{LE})] \end{Bmatrix} \tag{16}$$

where

$$\Delta A_n = A_n^{top} - A_n^{bottom};\ \Delta B_n = B_n^{top} - B_n^{bottom} \tag{17}$$

Formulation of this method for inverse design of three-dimensional aerodynamic shapes is a straightforward extension (Dulikravich and Baker, 1999a; 1999b).

1.4 Determination of Number, Sizes, Locations, and Shapes of Internal Coolant Flow Passages

During the past 15 years, our research team has been developing a unique inverse shape design methodology and accompanying software which allows a thermal system designer to determine the minimum number and correct sizes, shapes, and locations of coolant passages in arbitrarily-shaped internally-cooled configurations (Dulikravich, 1988; Dulikravich and Martin, 1996). The designer needs to specify both the desired temperatures and heat fluxes on the hot surface, and either temperatures or convective heat coefficients on the guessed internal coolant passage walls. The designer must also provide an initial guess of the total number, sizes, shapes, and locations of the coolant flow passages. Afterwards, the design process uses a constrained optimization algorithm to minimize the difference between the specified and computed hot surface heat fluxes by automatically relocating, resizing, reshaping and reorienting the initially-guessed coolant passages. All unnecessary coolant flow passages are reduced to a very small size and eliminated while honoring the specified minimum distances between the neighboring passages and between any passage and the thermal barrier coating if such exists. This type of computer code is highly economical, reliable, and geometrically flexible, if it utilizes the boundary element method (BEM) instead of finite element or finite difference method for the thermal field analysis. The BEM does not require generation of the interior grid and it is non-iterative. Thus, the method is computationally efficient and robust. The resulting shapes of coolant passages are smooth, and easily manufacturable. The methodology has been successfully demonstrated on coated and non-coated turbine blade airfoils, scramjet combustor struts, and three-dimensional coolant passages in the walls of rocket engine combustion chambers and axial gas turbine blades (Dulikravich and Martin, 1997).

1.5 Interior Void and Crack Shape Determination

The inverse determination of locations, sizes, and shapes of unknown interior voids subject to over-specified stress-strain outer surface field is a common inverse design problem in elasticity (Bezera and Saigal, 1993). Utilizing surface thermal boundary conditions (Dulikravich and Martin, 1993)) can also solve the void detection problem. The typical approach is to formulate a sum of least squares differences in the surface values of given and computed stresses or deformations (or temperatures or fluxes) for a guessed configuration of voids. This cost function is then minimized using any of the standard optimization algorithms by perturbing the number, sizes, shapes, and locations of the guessed voids. The process is identical to the already described inverse design of coolant flow passages subject to over-specified surface thermal conditions.

It should be pointed out that this approach to inverse detection of interior cavities and cracks could generate interior configurations that are non-unique.

2. BOUNDARY CONDITIONS DETERMINATION

A very common practical problem in any field theory is determination of the unknown boundary and initial conditions. Here, we will focus only on boundary conditions determination.

2.1 Determination of Steady Boundary Temperatures and Heat Fluxes

Determination of unknown steady thermal boundary conditions when neither temperature nor heat flux data are available on certain boundaries, is another common class of inverse problem. These unknown boundary conditions can be found if both temperature and heat flux are available on some other, more accessible boundaries or at a finite number of points within the domain. When using a BEM algorithm, if at all four vertices designated with subscripts 1, 2, 3, and 4 of a quadrilateral computational grid cell the heat sources pi are known, at two vertices both temperature and heat flux are known, while at the remaining two vertices neither temperature or heat flux is known, the boundary integral equation becomes (Martin and Dulikravich, 1997)

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \tilde{h}_{13} & \tilde{h}_{14} \\ \tilde{h}_{21} & \tilde{h}_{22} & \tilde{h}_{23} & \tilde{h}_{24} \\ \tilde{h}_{31} & \tilde{h}_{32} & \tilde{h}_{33} & \tilde{h}_{34} \\ \tilde{h}_{41} & \tilde{h}_{42} & \tilde{h}_{43} & \tilde{h}_{44} \end{bmatrix} \begin{Bmatrix} \overline{\Theta}_1 \\ \Theta_2 \\ \overline{\Theta}_3 \\ \Theta_4 \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \begin{Bmatrix} \overline{q}_1 \\ q_2 \\ \overline{q}_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} \tag{18}$$

Notice that [h] and [g] matrices depend on geometric relations and the configuration is known. If all of the unknowns are moved to the right-hand side, while all of the known thermal quantities are moved to the left, the result is

$$\begin{bmatrix} \tilde{h}_{12} & -g_{12} & \tilde{h}_{14} & -g_{14} \\ \tilde{h}_{22} & -g_{22} & \tilde{h}_{24} & -g_{24} \\ \tilde{h}_{32} & -g_{32} & \tilde{h}_{34} & -g_{34} \\ \tilde{h}_{42} & -g_{42} & \tilde{h}_{44} & -g_{44} \end{bmatrix} \begin{Bmatrix} \Theta_2 \\ q_2 \\ \Theta_4 \\ q_4 \end{Bmatrix} = \begin{bmatrix} -\tilde{h}_{11} & g_{11} & -\tilde{h}_{13} & g_{13} \\ -\tilde{h}_{21} & g_{21} & -\tilde{h}_{23} & g_{23} \\ -\tilde{h}_{31} & g_{31} & -\tilde{h}_{33} & g_{33} \\ -\tilde{h}_{41} & g_{41} & -\tilde{h}_{43} & g_{43} \end{bmatrix} \begin{Bmatrix} \overline{\Theta}_1 \\ \overline{q}_1 \\ \overline{\Theta}_3 \\ \overline{q}_3 \end{Bmatrix} + \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} \tag{19}$$

Since the entire right-hand side is known, it may be reformulated as a vector of known quantities, {**F**}. The left-hand side remains in the form [**A**]{**X**}. Additional equations may be added to this equation set if, for example, temperature measurements are known at certain locations within the domain.

In general, the geometric coefficient matrix [**A**] will be non-square and highly ill conditioned. Most matrix solvers will not work well enough to produce a correct solution. Singular Value Decomposition (SVD) methods (Press et al., 1992), are widely used in solving most linear least squares problems of this type. Thus, by using an SVD type algorithm it is possible to solve for the unknown surface temperatures and heat fluxes very accurately and non-iteratively.

2.2 Determination of Steady Convective Heat Transfer Coefficient Distribution without CFD Computations

Accurate values of the convective heat transfer coefficients are difficult to obtain experimentally, because their values depend strongly on at least twelve variables or eight non-dimensional groups. Rather than trying to evaluate the surface variation of the convective heat transfer coefficient it is possible to treat the heat convection coefficient determination problem as an ill-posed boundary value problem of pure heat conduction in the solid in contact with the moving fluid. Here, no thermal data is assumed available on parts of the boundary exposed to a moving fluid. This approach is capable of utilizing over-determined thermal measurements involving temperatures and heat fluxes on other boundaries or inside the solid where they are accessible.

The equation for the heat flux from the Robin boundary condition should be added to the linear BEM system governing the heat conduction problem in the solid that is in contact with the moving fluid. The unknown temperatures are then factored together with the other nodal temperatures appearing on the left-hand side of the BEM matrix equation set. After the ill-conditioned coefficient matrix [**A**] has been inverted using the SVD algorithm, the unknown boundary values of T and Q can be obtained from $\{X\} = [A]^{-1}\{F\}$. Once these thermal boundary values are determined on the boundary, the convective heat transfer coefficients can be determined from

$$h_{conv} = -k \frac{\partial T}{\partial n}\Big|_{\Gamma_{conv}} \Big/ \left(T\big|_{\Gamma_{conv}} - T_{amb} \right) \tag{20}$$

Here, T_{amb} is considered as known. The computed temperature field and the computed convective heat transfer coefficients on the boundary indicate increase in accuracy with the increased amount of over-specified data and the decrease in distance, b, between the over-specified and unspecified boundaries (Martin and Dulikravich, 1998). When repeated for a variety of practical Biot numbers ($B_i = h_{conv}$ b/k), this method was found to be reliable, and very fast, allowing realistic values of h_{conv} to be predicted in a few seconds on a standard PC.

2.3 Determination of Unsteady Thermal Boundary Conditions

In many problems involving unsteady cooling or heating of arbitrarily shaped objects it is often desirable to maintain a specified local cooling or heating rate in some parts of the solid object. The freezing of organs for transplant surgery is one example of such an inverse problem. This can be achieved by determining the appropriate time-variation of temperature and heat flux at every point of the walls of a cooling or heating container that will maintain the desired cooling or heating rate at the desired interior points. The local surface temperature of the cooling container should be continuously adjusted in time in order to maintain the specified local prescribed cooling rates throughout the object. To implement this at every instant of time, the container wall circumferential temperature variation was approximated (Dulikravich, 1988) using a Chebyshev polynomial in terms of the scaled circumferential angle. The coefficients of the polynomial were adjusted iteratively in order to maintain the desired cooling rates inside the object.

The process starts by specifying an initial wall temperature distribution and deducing the corresponding polynomial coefficients. These will be the initial values for the coefficients. Next, the transient temperature values are computed in the entire domain subject to the initial wall temperature distribution. From this, the local cooling rates are computed at a number of specified points inside the domain. A normalized cost function can then be formed as a sum of least squares of deviations of the computed and the specified local cooling rates. The new temperature distribution on the walls of the container is determined by minimizing the cost function at the next time step during the cooling process. Thus, the desired cooling rates are achieved throughout the object by determining the polynomial coefficients representing the proper variation of container wall temperatures at each instant of time.

2.4 Determination of Boundary Stresses and Deformations

An elastostatic problem is well-posed when the geometry of the general multiply-connected object is known and either displacement vectors, **u**, or surface traction vectors, **p**, are specified everywhere on the surface of the object. The elastostatic problem becomes ill posed when either: a) a part of the object's geometry is not known or b) when both **u** and **p** are unknown on certain parts of the surface. Both types of inverse problems can be solved only if additional information is provided. This information should be in the form of over-specified boundary conditions where both **u** and **p** are simultaneously provided at least on certain surfaces of the body. Using the BEM formulation, a system of algebraic equation can be formed that is similar to Eq. 18 in the case of heat conduction inverse boundary condition determination problems. Notice that each of the entries in the [h] and [g] matrices is a 2x2 sub-matrix in the case of a two-dimensional elasticity. Additional equations may be added to the equation set if **u** measurements are known at locations within the solid in order to enhance the accuracy of the inverse steady boundary condition determination algorithm. The equation system can be rearranged similar to Eq. 19 and solved non-iteratively using an SVD type algorithm (Martin et al., 1994).

3. SOURCES DETERMINATION

Many field quantities can be generated by either continuously spatially varying or discretely distributed sources of those field quantities. Determination of these continuously distributed or discretely distributed quantities is often of significant practical interest.

3.1 Determination of Continuous Heat Source Distribution

A standard test case for any such inverse algorithm is finding the internal heat generation function distribution when provided with over-specified thermal boundary conditions. We used (Martin and Dulikravich, 1996) an annular disk geometry with axisymmetric boundary conditions, $T_{outer} = T_{inner} = 0$ and a constant value of the heat source function. This well-posed problem has an analytic solution. These analytical values of heat fluxes were then used as the over-specified boundary conditions on the outer and inner circular boundaries in order to predict the value of the heat generation field. When the annular domain was discretized with quadrilateral cells circumferentially, having only one cell between the outer and inner circular boundaries, the heat generation field was predicted with an average error less than 0.01%. Similar results were found when the heat generation field was linearly varying with radius.

But, when the domain was discretized with two or more radial rows of quadrilateral cells, the results produced errors that were, at worst, in error by about 30%. This is because the assembled BEM matrix had at least twice as many unknowns as it had equations. The results were significantly improved whenever internal temperature measurements were included in the analysis. For example, when the domain was discretized with two rows of quadrilateral cells, an addition of a single row of nine known internal temperatures produced results which averaged an error of less than 0.1%.

Further results have shown that whenever the temperature field is entirely known everywhere in the domain, the resulting solution matrix is both square and well conditioned. After inversion of this matrix, the unknown heat source vector can be found with an accuracy comparable to the well-posed (forward) problem, where this vector is known and temperature field is the objective of the computation (Martin and Dulikravich, 1996).

3.2 Determination of Electric Dipoles in Electro-Cardiography

It is important to recognize that inverse BEM formulation is especially suitable for the detection of point-wise, isolated sources like in the ill-conditioned inverse problem of electro-cardiography (Bates, 1997). The accuracy of a variety of the existing techniques for inverse electro-cardiography is still very low since these problems result in highly ill conditioned systems of equations. Concentric spheres with centrally located multiple electric dipoles were used to simulate a heart and a torso and to evaluate the accuracy of the inverse BEM algorithm. The objective was to determine the strength of each of the dipoles that generates the measured electric potential on the surface of the torso. Results indicate that the inverse BEM technique provides solutions of comparable or higher accuracy with less computational time than other techniques (Bates, 1997). But, they also show that equivalent cardiac source models with large numbers of dipoles are still unreliable for computation of the inverse problems of this type due to uniqueness considerations. That is, practically the same distribution of the electric potential on the torso can be generated by more than one possible combination of numbers, strengths, and orientations of the electric dipoles in the heart.

4. PHYSICAL PROPERTY DETERMINATION

An increasingly important application of inverse methodology is determination of physical properties (thermal conductivity, electric conductivity, specific heat, thermal diffusivity, viscosity, magnetic permitivity, etc.) of the media. These properties could depend on certain field variables (temperature, pressure, density, frequency, etc.). Moreover, standards and regulations require that certain physical properties can be evaluated experimentally only by testing a specifically shaped, sized, and otherwise prepared material sample. Obviously, many applications do not allow the destruction of an object in order to extract such a sample. Thus, inverse determination of the physical properties is very popular in the non-destructive evaluation (NDE) community.

4.1 Determination of Temperature-Dependent Thermal Conductivity

This represents an inverse numerical procedure that differs substantially from the typical iterative approaches. It will be assumed that measured values of heat fluxes (or convection heat transfer coefficients) are available everywhere on the surface of an arbitrarily shaped solid. Kirchhoff’s transformation is then used to convert the governing steady heat conduction equation into a linear boundary value problem that can be solved for the unknown Kirchhoff’s heat functions on the boundary using the BEM. Given several boundary temperature measurements, these heat functions are then inverted to obtain thermal conductivity at the points where the over-specified temperature measurements were taken (Martin and Dulikravich, 1997).

The experimental part of this inverse method requires thermocouples and heat flux probes placed only on the surface of an arbitrarily shaped and sized specimen. Thus, this method is non-intrusive and directly applicable to field testing since special test specimens do not need to be manufactured. For steady-state problems, only one of each measurement device is needed for this methodology to work. This method could still use temperature measurements at isolated interior points if additional accuracy is desired. The method is inherently multi-dimensional and allows for temperature gradients in the test specimen.

The present method does not require that experimentally measured surface temperatures must be in equal temperature intervals. The present method also allows that convective heat transfer coefficients can be used instead of heat flux boundary conditions. This algorithm also accepts experimentally measured temperatures having same value, but measured at different boundary points.

Several different inversion procedures were attempted, including regularization, finite differencing, and least squares fitting with basis functions. The program was very accurate when the data was without error, and it did not excessively amplify input temperature measurement errors when those errors were less than 1-5% standard deviation. The program was found to be less sensitive to measurement errors in heat fluxes than to errors in temperatures. The accuracy of the algorithm was greatly increased with the use of *a priori* knowledge about the thermal conductivity basis functions.

It should be pointed out that in all applications and formulations that are briefly outlined in this paper, the inverse application of the BEM results in errors that are of the same order of magnitude as the errors in the over-specified boundary conditions (Martin and Dulikravich, 1996; 1997; 1999).

5. SIMULTANEOUS SOLUTION OF THERMO-ELASTICITY INVERSE PROBLEMS

The equations governing steady heat conduction and linear elasticity can be discretized by using a Galerkin's finite element method. For inverse problems, it is possible after a series of algebraic manipulations, to transform the original system of equations into a system that enforces the over-specified boundary conditions and includes the unknown boundary conditions as a part of the unknown solution vector (Dennis and Dulikravich, 1998). The resulting systems of equations will remain sparse, but will become unsymmetrical and possibly rectangular depending on the ratio of the number of known to unknown boundary conditions.

Three regularization methods were applied separately to the solution of the systems of equations in attempts to increase the method's tolerance for measurement errors in the over-specified boundary conditions.

The first method of regularization uses a constant damping parameter over the entire domain. This method can be considered the traditional Tikhonov method. The penalty term being minimized in this case is the square of the L_2 norm of the solution vector. Minimizing this penalty term will ultimately drive each component of the solution vector to zero, thus completely destroying the real solution.

The second method of regularization uses a constant damping parameter only for equations corresponding to the unknown boundary values since the largest errors occur at the boundaries where the temperatures, fluxes, stresses, and deformations are unknown.

The third method uses Laplacian smoothing of only on the boundaries where the boundary conditions are unknown. A penalty term could be constructed such that curvature of the solution on the unspecified boundary is minimized along with the residual.

In general, the resulting FEM systems for the inverse thermo-elastic problems are sparse, non-symmetric, and often rectangular rather than square. These properties make the process of finding a solution to the system very challenging. Three approaches will be discussed here.

The first is to normalize the equations by multiplying both sides by the matrix transpose and solve the resulting square system with common sparse solvers. The resulting normalized system is less sparse than the original system, but it is square, symmetric, and positive definite. It is typically solved with a direct method (Cholesky or LU factorization) or with an iterative method (preconditioned Krylov subspace). Disadvantages are computation expense of matrix multiplication, the large in-core memory requirements, and the round-off error incurred during the matrix multiplication.

A second approach is to use iterative methods suitable for unsymmetrical and least square problems. One such method is the LSQR method, which is an extension of the well-known conjugate gradient (CG) method. The LSQR method and other similar methods such as the conjugate gradient for least squares (CGLS) solve the normalized system, but without explicit matrix multiplication. However, convergence rates of these methods depend strongly on the condition number of the normalized system.

The third approach is to use a non-iterative method for non-symmetrical and least square problems such as QR factorization or SVD. However, sparse implementations of QR or SVD solvers are needed to reduce the in-core memory requirements for the inverse finite element problems.

All three sparse matrix solvers performed well for test cases with relatively small number of variables. The QR factorization was found to provide the highest accuracy in the shortest amount of computing time. However, it failed for larger problems where the number of grid points was greater than about 2000. This is most likely due to the instability of the QR algorithm when dealing with systems with high condition numbers. Applying small amounts of regularization to the sparse matrix eliminated the instability. The CG method applied to the normalized equations worked well for problems with less than 100 nodes. When regularization was applied to the sparse matrix, the CG convergence improved significantly, but the QR factorization was still much faster. The CGLS and LSQR methods were found to be slow for problems with more than 500 nodes, but were able to provide better solutions than those obtained with the CG (Dennis and Dulikravich, 1998).

6. SUMMARY AND RECOMMENDATIONS

We have sketched a number of concepts for achieving a solution of seemingly unsolvable (ill-posed) problems. We have often referred to the use of BEM because of its unique abilities to propagate the information from the boundaries throughout the domain without the need for iterations. However, FEM is expected to become a method of choice for realistic multidisciplinary three-dimensional problems. The pressing issue is further improvements in computational efficiency, accuracy, and especially reliability, are warranted for the matrix solution algorithms (especially for sparse matrices resulting from FEM) dealing with highly singular matrices. The future research on methods for the solution of inverse problems should focus on those methods that are applicable to arbitrary multiply connected three-dimensional domains, unsteady problems, and especially multidisciplinary problems.

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