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A FULLY NON-LINEAR THEORY OF ELECTRO-MAGNETO-HYDRODYNAMICS

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ABSTRACT

A number of analytical models exist for both electro-hydrodynamics (EHD), the study of fluid flows containing electric charges under the influence of an electric field and negligible magnetic field, and magneto-hydrodynamics (MHD), the study of fluid flows containing no free electric charges that are under the influence of a magnetic field and no electric field. At the present, there are no practical yet consistent models for the combined electro-magneto-hydrodynamic (EMHD) effects which most often occur in actual situations. This work represents an attempt to develop such a fully consistent analytical model for multi-dimensional, steady and unsteady, compressible and incompressible flows of electrically conducting fluids under the simultaneous or separate influence of externally applied and internally generated steady or unsteady electric and magnetic fields. The approach is based on the fundamental laws of continuum mechanics and thermodynamics with all assumptions clearly stated and consistently applied. The resulting second order EMHD model allows for non-linear polarization and magnetization of the medium. The new model is therefore superior to the existing EMHD models and represents a tractable set of equations suitable for detailed numerical discretization and integration.

NOMENCLATURE

$\underline{B} = \mu_0(\underline{H}+\underline{M})$	magnetic flux density, kg A^{-1} s^{-2}
B_0	reference value of $\underline{B}$, kg A^{-1} s^{-2}
$c = 3\times10^8$	speed of light in vacuum, m s^{-1}
C_p	specific heat at constant pressure, K^{-1} m^2 s^{-2}
$\underline{\underline{d}} = \frac{1}{2}[(\nabla\underline{v} + (\nabla\underline{v})^t]$	average rate of deformation tensor, s^{-1}
$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$	material derivative, s^{-1}
$\underline{D} = \varepsilon_0\underline{E} + \underline{P}$	electric flux density, A s m^{-2}
$\underline{E}$	electric field intensity, kg m s^{-3} A^{-1}
E_0	reference value of $\underline{E}$, kg m s^{-3} A^{-1}
$\hat{\underline{E}} = \underline{E} + \underline{v}\times\underline{B}$	electromotive intensity, kg m s^{-3} A^{-1}
$\underline{f}$	mechanical body force per unit mass, m s^{-2}
$\underline{F}^{em}$	electromagnetic body force per unit volume, kg m^{-2} s^{-2}
g	acceleration due to gravity, m s^{-2}
Q_h	heat source per unit volume, kg m^{-1} s^{-3}
$\underline{H}$	magnetic field intensity, A m^{-1}
$\underline{J} = \underline{J}_c + q_e\underline{v}$	electric current density, A m^{-2}
$\underline{J}_c$	electric conduction current, A m^{-2}
L	reference length, m
$\underline{M}$	total magnetization per unit volume, A m^{-1}
$\hat{\underline{M}} = \underline{M} + \underline{v}\times\underline{P}$	magnetomotive intensity, A m^{-1}
p	pressure, kg m^{-1} s^{-2}
$\underline{P}$	total polarization per unit volume, A s m^{-2}
q_e	total electric charge per unit volume, A s m^{-3}
$\underline{q}$	conduction heat flux, kg s^{-3}
s	entropy per unit mass, m^2 kg^{-1} K^{-1} s^{-2}
$\underline{\underline{t}} = -\varphi\underline{\underline{I}} + \underline{\underline{\tau}}$	Cauchy stress tensor, kg m^{-1} s^{-2}
T	absolute temperature, K
T_0	reference temperature, K
ΔT	reference temperature difference, K
$\hat{u}$	internal energy per unit mass, m^2 s^{-2}

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$\underline{v}$	fluid velocity, m s^{-1}
V	reference speed, m s^{-1}

Greek Symbols

$\beta = \varepsilon_0 \mu_0 V^2$	square of electromagnetic Mach number
ε	electric permittivity, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_0 = 8.854 \times 10^{-12}$	electric permittivity of vacuum, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_r = \varepsilon / \varepsilon_0$	relative electric permittivity
ρ	fluid density, kg m^{-3}
$\underline{\underline{\tau}}$	deviatoric part of stress tensor, kg m^{-1} s^{-2}
μ	magnetic permeability, kg m A^{-2} s^{-2}
θ	dimensionless temperature
$\mu_0 = 4\pi \times 10^{-7}$	magnetic permeability of vacuum, kg m A^{-2} s^{-2}
$\mu_r = \mu / \mu_0$	relative magnetic permeability
$\chi_e = \varepsilon_r - 1$	electric susceptibility
$\chi_m = \mu_r - 1$	magnetic susceptibility
$\Psi = \hat{u} - Ts - \underline{\hat{E}} \cdot \underline{P}/\rho$	generalized Helmholtz free energy per unit mass, m^2 s^{-2}

INTRODUCTION

The field studying fluid flows under the influence of the externally applied and internally generated electric and magnetic fields is often called electro-magneto-dynamics (EMD) of fluids (Hughes and Young, 1966), electro-magneto-fluid dynamics (EMFD) (Dulikravich and Lynn, 1995a; 1995b; 1997a; 1997b), electro-magneto-hydrodynamics (EMHD) (Dulikravich and Jing, 1996; 1997; Dulikravich, 1998), magneto-gas-dynamics (MGD) and plasma dynamics (Pai, 1963), or the electro-dynamics of material continua (Landau and Lifshitz, 1960; Eringen and Maugin, 1990a; 1990b). To reduce the complexity of the analytical models of this phenomenon, the analytical models have traditionally been simplified (Stuetzer, 1962) into Electro-Hydrodynamic (EHD) flows (Melcher, 1981) influenced only by an externally applied electric field acting upon electrically charged particles in the fluid, and Magneto-Hydrodynamic (MHD) flows (Sutton and Sherman, 1965) influenced only by an externally applied magnetic field without electric charges in the fluid. The existing EHD and MHD models often represent unacceptable oversimplifications of the actual combined electro-magnetic effects (Dulikravich and Lynn, 1995b; 1997b). More recently, rigorous continuum mechanical treatments of unified Electro-Magneto-Gasdynamic (EMGD) (Eringen and Maugin, 1990a) and EMHD flows (Eringen and Maugin, 1990b; Wineman and Rajagopal, 1995; Dulikravich and Lynn, 1995a; 1997a; Dulikravich and Jing, 1996; Dulikravich 1998) have been developed. These approaches are limited to non-relativistic, quasi-static, or relatively low frequency phenomenon (Bergman, 1962; Marcinkowski, 1992; Lakhtakia, 1993). The existing EMGD model is extremely complex and requires a large number of physical properties of the fluid, most of which are still unknown. Even in the case of EMHD (incompressible fluids under the influence of combined electric and magnetic fields) the existing models are not simple and not even fully consistent with the general EMGD model. Dulikravich and Jing (1996; 1997) have shown that a compact vector form of the unified EMGD system can be written as a combination of Maxwell's electro-magnetic subsystem and the Navier-Stokes fluid flow subsystem. Nevertheless, their version of the EMGD and especially of EMHD is not fully consistent with the most general version obtained by Eringen and Maugin (1990a; 1990b). In addition, the inconsistent models allow only for linear polarization and linear magnetization of the fluid.

The objective of this work is to summarize the most general EMGD analytical model, develop a rational second order approximation to the EMGD model, and finally to develop a fully consistent EMHD model that should supercede the existing inconsistent EMHD model (Dulikravich, 1998) and be acceptable for numerical discretization and integration. The consistent simplification of the general EMGD model will be performed using non-dimensionalization of each term in the governing equations.

GENERAL SET OF BALANCE LAWS

The full system of equations governing electro-magneto-gasdynamic (EMGD) and electro-magneto-hydrodynamic (EMHD) flows consists of Maxwell's equations governing electro-magnetism, the Navier-Stokes equations governing fluid flow, and constitutive equations describing material behavior. Assuming a single-phase fluid and only one type of charged particles in the fluid, this set has a minimum of twelve partial differential equations (Dulikravich and Lynn, 1995a; 1995b; 1997a; 1997b) that contains 13 unknowns: ρ, q_e, T, p, and the three vector components of $\underline{v}$, $\underline{E}$, and $\underline{B}$, respectively. The thirteenth equation is the equation of state for the compressible fluid, while ρ is assumed constant for the incompressible flows.

The firm foundations of the EMGD theory were formulated by Eringen and Maugin (1990a; 1990b) and are based on continuum mechanics. The rigor with which the constitutive, force, and energy terms were derived leads to a model more complete and robust than any of those found in classical literature. Following their approach, the balance laws for the electro-magnetic field are represented by Maxwell's equations.

The Maxwell's equations for polarizable and magnetizable medium can be written in a vector operator form as follows:

$$\nabla \cdot \underline{D} = q_e \,, \quad \text{(Gauss' law)} \tag{1}$$

$$\nabla \cdot \underline{B} = 0 \,, \quad \text{(conservation of magnetic flux)} \tag{2}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \,, \quad \text{(Faraday's law)} \tag{3}$$

$$\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} + \underline{J} \,. \quad \text{(Ampere-Maxwell's law)} \tag{4}$$

This set of equations defines the divergence and curl of the electric and magnetic field, respectively. It is well known that a vector field can be completely determined if its divergence and curl are known. Note that the relations between flux density and field intensity vectors in the polarizable and magnetizable medium are

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} \,, \tag{5}$$

$$\underline{B} = \mu_0 (\underline{H} + \underline{M}) \,. \tag{6}$$

It should also be pointed out that strengths of polarization and magnetization strongly depend on the material and that they can be estimated from the constitutive equations. In addition, equation of conservation of electric charges is derived from a combination of Gauss' law and Ampere-Maxwell's law,

$$\frac{\partial q_e}{\partial t} + \nabla \cdot \underline{J} = 0, \tag{7}$$

It should be noted that this equation accounts for all types of charged species together since charge transport takes place by charge carrier motion and by charge jumping from one carrier to another.

The balance laws of the thermo-mechanical field are comprised of the conservation laws and the second law of thermodynamics. The conservation of mass is expressed by the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0. \tag{8}$$

Conservation of linear momentum with mechanical and electro-magnetic body forces requires that the momentum equation,

$$\rho \frac{D\underline{v}}{Dt} = \nabla \cdot \underline{\underline{t}} + \rho \underline{f} + \underline{F}^{em}, \tag{9}$$

be satisfied. The general form of the electro-magnetic body force per unit volume is expressed as (Eringen and Maugin, 1990a)

$$\begin{aligned} \underline{F}^{em} &= q_e \underline{E} + \underline{J} \times \underline{B} + \underline{P} \cdot \nabla \underline{E} + \underline{M} \cdot \nabla \underline{B} \\ &+ \nabla \cdot \left[\underline{v} \left(\underline{P} \times \underline{B} \right) \right] + \frac{\partial}{\partial t} \left(\underline{P} \times \underline{B} \right) \end{aligned} \tag{10}$$

while the surface force density (stress tensor) can be determined from the constitutive relations. The conservation of energy with internal heat sources can be represented by the energy equation, expressed as

$$\rho \frac{D\hat{u}}{Dt} = \underline{\underline{t}} : \underline{\underline{d}} + Q_h + \nabla \cdot \underline{q} + \rho \underline{\hat{E}} \cdot \frac{D(\rho^{-1} \underline{P})}{Dt} - \underline{\hat{M}} \cdot \frac{D\underline{B}}{Dt} + \underline{J}_c \cdot \underline{\hat{E}}. \tag{11}$$

The final balance law comes from the second law of thermodynamics. This is represented by the Clausius-Duhem (C-D) inequality

$$\rho \frac{Ds}{Dt} - \nabla \cdot \left(T^{-1} \underline{q} \right) - T^{-1} Q_h \geq 0. \tag{12}$$

By using the energy equation, and by introducing the generalized Helmholtz free energy,

$$\Psi = \hat{u} - Ts - \rho^{-1} \underline{\hat{E}} \cdot \underline{P}, \tag{13}$$

the C-D inequality can be rewritten as

$$\begin{aligned} \rho\gamma \equiv -\rho \left(\frac{D\Psi}{Dt} + s \frac{DT}{Dt} \right) + \underline{\underline{t}} : \underline{\underline{d}} + T^{-1} \underline{q} \cdot \nabla T \\ - \underline{P} \cdot \frac{D\underline{\hat{E}}}{Dt} - \underline{\hat{M}} \cdot \frac{D\underline{B}}{Dt} + \underline{J}_c \cdot \underline{\hat{E}} \geq 0. \end{aligned} \tag{14}$$

GENERAL CONSTITUTIVE EQUATIONS

In order to completely determine the electro-magnetic and thermo-mechanical fields, the balance laws must be supplemented by the constitutive equations since the number of unknowns is larger than the number of the balance equations. Given the mechanical body force and the internal heat source, additional information about the polarization and magnetization, stress tensor, electric conduction, and conduction heat transfer are required. The most general theory of constitutive equations relating the electro-magnetic and thermo-mechanical effects has been developed by Eringen and Maugin (1990a; 1990b), in which a continuum approach has been used. Since the second order model starts with the general nonlinear theory, some of the essentials will be reproduced here.

For simple rate-dependent, memory-independent, and isotropic fluids, field variables (scalar, vector, and tensor) can be represented as functions of $\underline{\underline{d}}, \underline{\hat{E}}, \underline{B}, \nabla T, T$, and ρ. Therefore, Ψ takes the form

$$\Psi = \Psi(\underline{\underline{d}}, \underline{\hat{E}}, \underline{B}, \nabla T, T, \rho). \tag{15}$$

Similar equations with the same arguments are valid for the stress tensor, electric conduction current vector, etc. Substitution of this form into the C-D inequality and the interpretation of the inequality yield the following restrictions on the constitutive equations:

$$\frac{\partial \Psi}{\partial \underline{\underline{d}}} = 0, \quad \frac{\partial \Psi}{\partial \nabla T} = 0, \quad s = -\frac{\partial \Psi}{\partial T}, \tag{16}$$

$$\underline{P} = -\rho \frac{\partial \Psi}{\partial \underline{\hat{E}}}, \quad \underline{\hat{M}} = -\rho \frac{\partial \Psi}{\partial \underline{B}}, \tag{17}$$

$$\rho\gamma \equiv \underline{\underline{\tau}} : \underline{\underline{d}} + \frac{1}{T} \underline{q} \cdot \nabla T + \underline{J}_c \cdot \underline{\hat{E}} \geq 0, \tag{18}$$

where

$$\underline{\underline{t}} = -\varphi \underline{\underline{I}} + \underline{\underline{\tau}}, \quad \varphi = \rho^2 \frac{\partial \Psi}{\partial \rho}. \tag{19}$$

The new C-D inequality, which states the irreversible part of entropy generation, can be valid only if the following condition

$$\underline{\underline{\tau}} = 0, \ \underline{J}_c = 0, \ \underline{q} = 0 \ \text{when} \ \underline{\underline{d}} = 0, \underline{\hat{E}} = 0, \nabla T = 0 \tag{20}$$

holds for the constitutive equations. The first two equations imply that Ψ is independent of $\underline{\underline{d}}$ and ∇T. In a material continuum, the Poynting vector depends on $\underline{\hat{E}}(\underline{r}, t) \cdot \underline{D}(\underline{r}, t)$ and $\underline{H}(\underline{r}, t) \cdot \underline{B}(\underline{r}, t)$. In assuming that Ψ depends on $\underline{\hat{E}}(t)$ and $\underline{B}(t)$ alone, the assumption of a medium with purely instantaneous response has been made (Lakhtakia and Weiglhofer, 1995). Then, the possible dependence of the free energy upon $\underline{\hat{E}}$ and $\underline{B}$ is only through its scalar invariants defined by,

$$I_1 = \underline{\hat{E}} \cdot \underline{\hat{E}}, \quad I_2 = \underline{B} \cdot \underline{B}, \quad I_3 = (\underline{\hat{E}} \cdot \underline{B})^2 . \tag{21}$$

Consequently, the free energy, entropy, polarization, and magnetization must take the following general form.

$$\Psi = \Psi(I_1, I_2, I_3, T, \rho), \quad s = -\partial\Psi/\partial t , \tag{22}$$

$$\underline{P} = -2\rho\left(\frac{\partial\Psi}{\partial I_1}\underline{\hat{E}} + \frac{\partial\Psi}{\partial I_3}\left(\underline{\hat{E}} \cdot \underline{B}\right)\underline{B}\right) \equiv \varepsilon_0\chi_e\underline{\hat{E}} + \lambda\left(\underline{\hat{E}} \cdot \underline{B}\right)\underline{B} , \tag{23}$$

$$\underline{\hat{M}} = -2\rho\left(\frac{\partial\Psi}{\partial I_2}\underline{B} + \frac{\partial\Psi}{\partial I_3}\left(\underline{\hat{E}} \cdot \underline{B}\right)\underline{\hat{E}}\right) \equiv \frac{\chi_B}{\mu_0}\underline{B} + \lambda\left(\underline{\hat{E}} \cdot \underline{B}\right)\underline{\hat{E}} . \tag{24}$$

The electric and magnetic susceptibilities, χ_e and χ_B are defined by this equation. The modified hydrostatic pressure is also represented by the equation

$$\varphi = \varphi(I_1, I_2, I_3, T, \rho) = \rho^2\partial\Psi/\partial\rho . \tag{25}$$

Using similar reasoning, the symmetric, electro-magneto-thermo-mechanical stress tensor for a non-linear fluid can be expressed as (similar to Eringen and Maugin, 1990a, pp.177-178)

$$\begin{aligned}
\underline{\underline{\tau}} = {} & \alpha_0\underline{\underline{I}} + \alpha_1\underline{\underline{d}} + \alpha_2\underline{\underline{d}}^2 + \alpha_3\underline{\hat{E}} \otimes \underline{\hat{E}} + \alpha_4\underline{B} \otimes \underline{B} + \alpha_5\nabla T \otimes \nabla T \\
& + \alpha_6\left(\underline{\hat{E}} \otimes \underline{\underline{d}} \cdot \underline{\hat{E}}\right)_s + \alpha_7\left(\underline{\hat{E}} \otimes \underline{\underline{d}}^2 \cdot \underline{\hat{E}}\right)_s + \alpha_8\left(\nabla T \otimes \underline{\underline{d}} \cdot \nabla T\right)_s \\
& + \alpha_9\left(\nabla T \otimes \underline{\underline{d}}^2 \cdot \nabla T\right)_s + \alpha_{10}\left(\underline{\underline{d}} \cdot \underline{\underline{W}}\right)_s + \alpha_{11}\underline{\underline{W}} \cdot \underline{\underline{d}} \cdot \underline{\underline{W}} \\
& + \alpha_{12}\left(\underline{\underline{d}}^2 \cdot \underline{\underline{W}}\right)_s + \alpha_{13}\left(\underline{\underline{W}} \cdot \underline{\underline{d}} \cdot \underline{\underline{W}}^2\right)_s + \alpha_{14}\left(\underline{\hat{E}} \otimes \nabla T\right)_s \\
& + \alpha_{15}\left(\underline{\underline{W}} \cdot \underline{\hat{E}} \otimes \underline{\underline{W}} \cdot \underline{\hat{E}}\right) + \alpha_{16}\left(\underline{\hat{E}} \otimes \underline{\underline{W}} \cdot \underline{\hat{E}}\right)_s \\
& + \alpha_{17}\left(\underline{\underline{W}} \cdot \underline{\hat{E}} \otimes \underline{\underline{W}}^2 \cdot \underline{\hat{E}}\right)_s + \alpha_{18}\left(\underline{\underline{W}} \cdot \nabla T\right) \otimes \left(\underline{\underline{W}} \cdot \nabla T\right) \\
& + \alpha_{19}\nabla T \otimes \left(\underline{\underline{W}} \cdot \nabla T\right) + \alpha_{20}\left(\left(\underline{\underline{W}} \cdot \nabla T\right) \otimes \left(\underline{\underline{W}}^2 \cdot \nabla T\right)\right)_s \\
& + \alpha_{21}\left(\underline{\underline{d}} \cdot \left(\underline{\hat{E}} \otimes \nabla T - \nabla T \otimes \underline{\hat{E}}\right)\right)_s \\
& + \alpha_{22}\left(\underline{\underline{W}} \cdot \left(\underline{\hat{E}} \otimes \nabla T - \nabla T \otimes \underline{\hat{E}}\right)\right)_s .
\end{aligned} \tag{26}$$

In this expression, $\underline{\underline{W}} = W_{kl} = \varepsilon_{klm}B_m$, while the subscript s indicates symmetrization. Electric conduction current in its most general case can be expressed as (Eringen and Maugin, 1990a, pp.161-162)

$$\begin{aligned}
\underline{J}_c = {} & \sigma_1\underline{\hat{E}} + \sigma_2\underline{\underline{d}} \cdot \underline{\hat{E}} + \sigma_3\underline{\underline{d}}^2 \cdot \underline{\hat{E}} + \sigma_4\nabla T + \sigma_5\underline{\underline{d}} \cdot \nabla T \\
& + \sigma_6\underline{\underline{d}}^2 \cdot \nabla T + \sigma_7\underline{\hat{E}} \times \underline{B} + \sigma_8\left(\underline{B} \cdot \underline{\hat{E}}\right)\underline{B} \\
& + \sigma_9\left(\underline{\underline{d}} \cdot (\underline{\hat{E}} \times \underline{B}) - (\underline{\underline{d}} \cdot \underline{\hat{E}}) \times \underline{B}\right) + \sigma_{10}\nabla T \times \underline{B} \\
& + \sigma_{11}\left(\underline{B} \cdot \nabla T\right)\underline{B} + \sigma_{12}\left(\underline{\underline{d}} \cdot (\nabla T \times \underline{B}) - (\underline{\underline{d}} \cdot \nabla T) \times \underline{B}\right) .
\end{aligned} \tag{27}$$

Conduction heat flux in its most general case can be expressed as

$$\begin{aligned}
\underline{q} = {} & \kappa_1\underline{\hat{E}} + \kappa_2\underline{\underline{d}} \cdot \underline{\hat{E}} + \kappa_3\underline{\underline{d}}^2 \cdot \underline{\hat{E}} + \kappa_4\nabla T + \kappa_5\underline{\underline{d}} \cdot \nabla T \\
& + \kappa_6\underline{\underline{d}}^2 \cdot \nabla T + \kappa_7\underline{\hat{E}} \times \underline{B} + \kappa_8\left(\underline{B} \cdot \underline{\hat{E}}\right)\underline{B} \\
& + \kappa_9\left(\underline{\underline{d}} \cdot (\underline{\hat{E}} \times \underline{B}) - (\underline{\underline{d}} \cdot \underline{\hat{E}}) \times \underline{B}\right) + \kappa_{10}\nabla T \times \underline{B} \\
& + \kappa_{11}\left(\underline{B} \cdot \nabla T\right)\underline{B} + \kappa_{12}\left(\underline{\underline{d}} \cdot (\nabla T \times \underline{B}) - (\underline{\underline{d}} \cdot \nabla T) \times \underline{B}\right) .
\end{aligned} \tag{28}$$

The scalar coefficients of the tensors and vectors used in these expressions are functions of the joint scalar invariants of $\underline{\underline{d}}, \underline{\hat{E}}, \underline{B}, \nabla T$, and T, ρ. In other words,

$$\begin{aligned}
& \alpha_i = \alpha_i(I_n, T, \rho), \\
& \sigma_j = \sigma_j(I_n, T, \rho), \quad \kappa_j = \kappa_j(I_n, T, \rho) \\
& \left(i = 0,1,\ldots,22;\ j = 1,2,\ldots,12;\ n = 1,2,\ldots27\right) .
\end{aligned} \tag{29}$$

An irreducible set of twenty seven joint scalar invariants, including the three already defined, is defined as

$$\begin{aligned}
& I_4 = \mathrm{tr}(\underline{\underline{d}}) = \nabla \cdot \underline{v} , \quad I_5 = \mathrm{tr}(\underline{\underline{d}}^2) , \\
& I_6 = \mathrm{tr}(\underline{\underline{d}}^3) , \quad I_7 = \nabla T \cdot \nabla T , \quad I_8 = \underline{\hat{E}} \cdot \underline{\underline{d}} \cdot \underline{\hat{E}} , \\
& I_9 = \underline{\hat{E}} \cdot \underline{\underline{d}}^2 \cdot \underline{\hat{E}} , \quad I_{10} = \nabla T \cdot \underline{\underline{d}} \cdot \nabla T , \\
& I_{11} = \nabla T \cdot \underline{\underline{d}}^2 \cdot \nabla T , \quad I_{12} = \underline{B} \cdot \underline{\underline{d}} \cdot \underline{B} , \\
& I_{13} = \underline{B} \cdot \underline{\underline{d}}^2 \cdot \underline{B} , \quad I_{14} = \underline{B} \cdot \left(\left(\underline{\underline{d}} \cdot \underline{B}\right) \times \left(\underline{\underline{d}}^2 \cdot \underline{B}\right)\right), \\
& I_{15} = \underline{\hat{E}} \cdot \nabla T , \quad I_{16} = \left(\underline{B} \cdot \nabla T\right)^2 , \quad I_{17} = \underline{\hat{E}} \cdot \underline{\underline{d}} \cdot \nabla T , \\
& I_{18} = \underline{\hat{E}} \cdot \underline{\underline{d}}^2 \cdot \nabla T , \quad I_{19} = \underline{\hat{E}} \cdot \left(\nabla T \times \underline{B}\right), \\
& I_{20} = \left(\underline{\hat{E}} \cdot \underline{B}\right)\left(\nabla T \cdot \underline{B}\right), \quad I_{21} = \underline{\hat{E}} \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \underline{\hat{E}}\right)\right), \\
& I_{22} = \nabla T \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \nabla T\right)\right), \quad I_{23} = \underline{\hat{E}} \cdot \left(\underline{B} \times \left(\underline{\underline{d}}^2 \cdot \underline{E}\right)\right), \\
& I_{24} = \nabla T \cdot \left(\underline{B} \times \left(\underline{\underline{d}}^2 \cdot \nabla T\right)\right), \quad I_{25} = \left(\underline{\hat{E}} \cdot \underline{B}\right)\underline{\hat{E}} \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \underline{B}\right)\right), \\
& I_{26} = \left(\nabla T \cdot \underline{B}\right)\nabla T \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \underline{B}\right)\right), \\
& I_{27} = \underline{\hat{E}} \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \nabla T\right)\right) + \nabla T \cdot \left(\underline{B} \times \left(\underline{\underline{d}} \cdot \underline{\hat{E}}\right)\right).
\end{aligned} \tag{30}$$

SECOND ORDER THEORY OF CONSTITUTIVE EQUATIONS

Because of the large number of joint invariants, the constitutive equations for the stress tensor, electric current, and heat flux vector are too complicated for practical applications. While the linear theory is relatively simple (Dulikravich, 1998), it is inconsistent and inappropriate for cases where nonlinear and/or cross effects are important. Hence, it is natural to seek a fully consistent nonlinear theory with minimal complication, which is called second order theory. It is based upon the assumption that the electro-magnetic fields, rate of shearing strain, and temperature gradient are small.

In the derivation of constitutive equations the following assumptions will be used. First, only the terms up to second order in $\underline{\underline{d}}$, $\hat{\underline{E}}$, $\underline{B}$, ∇T altogether will be retained. Second, terms of second and higher orders in $\underline{\underline{d}}$ will be neglected as in the case of conventional Newtonian fluids. The application of these two assumptions to the general form of the previous section makes the constitutive equations for Ψ, $\underline{P}$, $\hat{\underline{M}}$, and φ be simplified to

$$\Psi = \Psi(I_1, I_2, T, \rho) = \Psi_0 - \frac{1}{2\rho}\left(\varepsilon_0\chi_e I_1 + \frac{\chi_B}{\mu_o} I_2\right), \tag{31}$$

$$\begin{aligned}\varphi &= \rho^2\frac{\partial\Psi_o}{\partial\rho} - \frac{\varepsilon_o\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{\chi_e}{\rho}\right)I_1 - \frac{\rho^2}{2\mu_o}\frac{\partial}{\partial\rho}\left(\frac{\chi_B}{\rho}\right)I_2 \\ &\equiv \varphi_o + \varphi_e + \varphi_m ,\end{aligned} \tag{32}$$

$$\underline{P} = \varepsilon_o\chi_e\hat{\underline{E}}, \tag{33}$$

$$\hat{\underline{M}} = \frac{\chi_B}{\mu_o}\underline{B}, \tag{34}$$

which indicate a medium with purely instantaneous response (Lakhtakia and Weiglhofer, 1995). Here, Ψ_o, χ_e, χ_B and φ_o, φ_e/I_1, φ_m/I_2 are functions of T and ρ. In the same way the Cauchy stress tensor, electric conduction current, and conduction heat flux can be consistently simplified as follows:

$$\underline{\underline{t}} = -\varphi\underline{\underline{I}} + \underline{\underline{\tau}}, \tag{35}$$

$$\begin{aligned}\underline{\underline{\tau}} = &\left(\alpha_{00} + \alpha_{01}I_1 + \alpha_{02}I_2 + \alpha_{04}I_4 + \alpha_{07}I_7 + \alpha_{015}I_{15}\right)\underline{\underline{I}} + \\ &\alpha_1\underline{\underline{d}} + \alpha_3\hat{\underline{E}}\otimes\hat{\underline{E}} + \alpha_4\underline{B}\otimes\underline{B} + \alpha_5\nabla T\otimes\nabla T \\ &+ \alpha_{10}\left(\underline{\underline{d}}\cdot\underline{\underline{W}}\right)_S + \alpha_{14}\left(\hat{\underline{E}}\otimes\nabla T\right)_S ,\end{aligned} \tag{36}$$

$$\begin{aligned}\underline{J}_c = &(\sigma_1 + \sigma_{1b}I_4)\hat{\underline{E}} + \sigma_2\underline{\underline{d}}\cdot\hat{\underline{E}} + (\sigma_4 + \sigma_{4b}I_4)\nabla T \\ &+ \sigma_5\underline{\underline{d}}\cdot\nabla T + \sigma_7\hat{\underline{E}}\times\underline{B} + \sigma_{10}\nabla T\times\underline{B} ,\end{aligned} \tag{37}$$

$$\begin{aligned}\underline{q} = &(\kappa_1 + \kappa_{1b}I_4)\hat{\underline{E}} + \kappa_2\underline{\underline{d}}\cdot\hat{\underline{E}} + (\kappa_4 + \kappa_{4b}I_4)\nabla T \\ &+ \kappa_5\underline{\underline{d}}\cdot\nabla T + \kappa_7\hat{\underline{E}}\times\underline{B} + \kappa_{10}\nabla T\times\underline{B} .\end{aligned} \tag{38}$$

Here, all the coefficients α, σ, κ are general functions of T and ρ. The possible restrictions imposed on these coefficients can be sought through the C-D inequality. First, the requirement

$$\underline{\underline{\tau}} = 0 \text{ when } \underline{\underline{d}} = 0,\ \hat{\underline{E}} = 0,\ \nabla T = 0 \tag{39}$$

results in

$$\alpha_{00} = \alpha_{02} = 0;\ \alpha_4 = 0. \tag{40}$$

And then, the non-negative irreversible entropy generation function can then be computed as

$$\begin{aligned}\rho\gamma = &\left(\alpha_1\underline{\underline{d}}:\underline{\underline{d}} + \alpha_{04}\left(\mathrm{tr}\left(\underline{\underline{d}}\right)\right)^2\right) + (\alpha_3 + \sigma_2)(\hat{\underline{E}}\cdot\underline{\underline{d}}\cdot\hat{\underline{E}}) \\ &+ \left[T^{-1}\kappa_1\left(\nabla T\cdot\nabla T\right) + \left(T^{-1}\kappa_4 + \sigma_4\right)\left(\hat{\underline{E}}\cdot\nabla T\right) + \sigma_1\left(\hat{\underline{E}}\cdot\hat{\underline{E}}\right)\right] \\ &+ (\alpha_5 + T^{-1}\kappa_2)(\nabla T\cdot\underline{\underline{d}}\cdot\nabla T) + \alpha_{10}\left(\underline{\underline{d}}\cdot\underline{\underline{W}}\right)_S : \underline{\underline{d}} \\ &+ \left\{(\alpha_{01} + \sigma_{1b})(\hat{\underline{E}}\cdot\hat{\underline{E}}) + (\alpha_{07} + T^{-1}\kappa_{1b})(\nabla T\cdot\nabla T)\right. \\ &\left.+ (\alpha_{015} + T^{-1}\kappa_{4b} + \sigma_{4b})\left(\hat{\underline{E}}\cdot\nabla T\right)\right\}\mathrm{tr}\left(\underline{\underline{d}}\right) \\ &+ (\alpha_{14} + T^{-1}\kappa_5 + \sigma_5)(\hat{\underline{E}}\cdot\underline{\underline{d}}\cdot\nabla T) \\ &+ (T^{-1}\kappa_{10} - \sigma_{10})\nabla T\cdot\left(\hat{\underline{E}}\times\underline{B}\right).\end{aligned} \tag{41}$$

In order that this function should be non-negative for any values of $\underline{\underline{d}}, \hat{\underline{E}}, \underline{B}, \nabla T$, the following relations among the physical properties must be satisfied

$$\begin{aligned}&\alpha_1 \ge 0,\quad \sigma_1 \ge 0,\quad \kappa_1 \ge 0,\quad 3\alpha_{04} + \alpha_1 \ge 0, \\ &\left(T^{-1}\kappa_4 + \sigma_4\right)^2 \le 4T^{-1}\kappa_1\sigma_1,\quad \alpha_3 + \sigma_2 = 0, \\ &\alpha_{07} + T^{-1}\kappa_{1b} = 0,\quad \alpha_{10} = 0,\quad \alpha_{01} + \sigma_{1b} = 0, \\ &T^{-1}\kappa_{10} - \sigma_{10} = 0,\quad \alpha_{015} + T^{-1}\kappa_{4b} + \sigma_{4b} = 0, \\ &\alpha_5 + T^{-1}\kappa_2 = 0,\quad \alpha_{14} + T^{-1}\kappa_5 + \sigma_5 = 0.\end{aligned} \tag{42}$$

The deviator part of the Cauchy stress tensor then becomes

$$
\begin{aligned}
\underline{\underline{\tau}} = &\left\{-\sigma_{1b}I_1 + \alpha_{04}I_4 - T^{-1}\kappa_{1b}I_7 - \left(T^{-1}\kappa_{4b} + \sigma_{4b}\right)I_{15}\right\}\underline{\underline{I}} \\
&+ \alpha_1\underline{\underline{d}} - \sigma_2\underline{\hat{E}} \otimes \underline{\hat{E}} - T^{-1}\kappa_2\nabla T \otimes \nabla T \\
&- \left(T^{-1}\kappa_5 + \sigma_5\right)\left(\underline{\hat{E}} \otimes \nabla T\right)_s .
\end{aligned}
\tag{43}
$$

There is also a slight change in the expression for electric conduction current into

$$
\begin{aligned}
\underline{J}_c = &\left(\sigma_1 + \sigma_{1b}I_4\right)\underline{\hat{E}} + \sigma_2\underline{\underline{d}} \cdot \underline{\hat{E}} + \left(\sigma_4 + \sigma_{4b}I_4\right)\nabla T \\
&+ \sigma_5\underline{\underline{d}} \cdot \nabla T + \sigma_7\underline{\hat{E}} \times \underline{B} + \frac{\kappa_{10}}{T}\nabla T \times \underline{B}
\end{aligned}
\tag{44}
$$

while that of the conduction heat flux remains unchanged.

CONSTITUTIVE EQUATIONS FOR INCOMPRESSIBLE FLOWS

In the case of an incompressible fluid, all terms related to $I_4 (= \mathrm{tr}(\underline{\underline{d}}) = \nabla \cdot \underline{v})$ are zero and physical properties do not depend on density. Constitutive equations are then given as

$$
\underline{P} = \varepsilon_o\chi_e\underline{\hat{E}} ,
\tag{45}
$$

$$
\underline{\hat{M}} = \frac{\chi_B}{\mu_o}\underline{B} ,
\tag{46}
$$

$$
\underline{\underline{t}} = -\varphi\underline{\underline{I}} + \underline{\underline{\tau}}
\tag{47}
$$

where

$$
\begin{aligned}
\varphi &= \rho^2\frac{\partial\Psi_o}{\partial\rho} + \frac{\varepsilon_o\chi_e}{2}\underline{\hat{E}} \cdot \underline{\hat{E}} + \frac{\chi_B}{2\mu_o}\underline{B} \cdot \underline{B} \\
&= p + p_e + p_m
\end{aligned}
\tag{48}
$$

and

$$
\underline{\underline{\tau}} = \alpha_1\underline{\underline{d}} - \sigma_2\underline{\hat{E}} \otimes \underline{\hat{E}} - \frac{\kappa_2}{T}\nabla T \otimes \nabla T - \left(\frac{\kappa_5}{T} + \sigma_5\right)\left(\underline{\hat{E}} \cdot \nabla T\right)_s .
\tag{49}
$$

The general expression for electric conduction current similarly reduces to

$$
\begin{aligned}
\underline{J}_c = &\sigma_1\underline{\hat{E}} + \sigma_2\underline{\underline{d}} \cdot \underline{\hat{E}} + \sigma_4\nabla T + \sigma_5\underline{\underline{d}} \cdot \nabla T \\
&+ \sigma_7\underline{\hat{E}} \times \underline{B} + \frac{\kappa_{10}}{T}\nabla T \times \underline{B}
\end{aligned}
\tag{50}
$$

while the general expression for conduction heat flux reduces to

$$
\begin{aligned}
\underline{q} = &\kappa_1\nabla T + \kappa_2\underline{\underline{d}} \cdot \nabla T + \kappa_4\underline{\hat{E}} + \kappa_5\underline{\underline{d}} \cdot \underline{\hat{E}} \\
&+ \kappa_7\nabla T \times \underline{B} + \kappa_{10}\underline{\hat{E}} \times \underline{B} .
\end{aligned}
\tag{51}
$$

Notice that in the case of EMHD, the following physical properties of the media can be either constants or functions of temperature only:

$$
\chi_e, \chi_B;\ \alpha_1;\ \sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7;\ \kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_{10} .
$$

Mechanical pressure, p, must be determined so that

$$
\nabla \cdot \underline{v} = 0
\tag{52}
$$

is satisfied everywhere in the flow-field. The following relations must also be satisfied according to C-D inequality.

$$
\alpha_1 \geq 0,\ \sigma_1 \geq 0,\ \kappa_1 \geq 0,\ \left(T^{-1}\kappa_4 + \sigma_4\right)^2 \leq 4T^{-1}\kappa_1\sigma_1 .
\tag{53}
$$

After neglecting terms of the order of β, magnetic field and magnetization vectors become

$$
\underline{H} = \frac{1}{\mu}\underline{B} + \varepsilon_p\underline{v} \times \underline{E} ,
\tag{54}
$$

$$
\underline{M} = \frac{1}{\mu_m}\underline{B} - \varepsilon_p\underline{v} \times \underline{E} .
\tag{55}
$$

For simplicity of notation, the following relations will be used

$$
\varepsilon_0\chi_e = \varepsilon_p = \varepsilon - \varepsilon_0 ,
\tag{56}
$$

$$
\frac{\chi_B}{\mu_0} = \frac{1}{\mu_m} = \frac{1}{\mu_0} - \frac{1}{\mu} = \frac{1}{\mu_0}\left(\frac{\chi_m}{1 + \chi_m}\right).
\tag{57}
$$

NON-LINEAR MODEL OF EMHD FLOWS

A full system of governing equations for the EMHD flows is described in this section by using the constitutive equations through the second order theory.

A slight modification of the conservation laws, the so-called Boussinesq approximation, is needed to be compatible with incompressible flows. In the Boussinesq approximation, the variation of density is kept only in the gravity force of the momentum equation. A linear dependence of density on temperature is assumed there. If the thermal buoyancy is the only mechanical body force acting on the fluid, the linear momentum equation can be generalized as:

$$
\frac{D\underline{v}}{Dt} = \rho^{-1}(\nabla \cdot \underline{\underline{t}} + \underline{F}^{em}) - g[1 - \alpha(T - T_0)]\underline{i_3} .
\tag{58}
$$

Here, α is the coefficient of thermal expansion of the fluid. It is a common practice to neglect the thermal buoyancy and to use $\alpha = 0$ in forced convection studies. The energy conservation equation for incompressible flows can also be written as

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \underline{\underline{t}} : \underline{\underline{d}} + Q_h + \nabla \cdot \underline{q} + \underline{\hat{E}} \cdot \frac{D\underline{P}}{Dt} - \underline{\hat{M}} \cdot \frac{D\underline{B}}{Dt} + \underline{J}_c \cdot \underline{\hat{E}} , \tag{59}$$

where

$$\frac{D\hat{u}}{Dt} = C_p \frac{DT}{Dt} - \rho^{-1} \frac{Dp}{Dt} \tag{60}$$

was used, as it is valid for the incompressible flows.

By substituting the constitutive relations for the polarization, magnetization, stress tensor, electric conduction current, and conduction heat flux into all the balance laws, we get the following system of governing equations:

Maxwell's equations:

$$\nabla \cdot \left(\varepsilon \underline{E} + \varepsilon_p \underline{v} \times \underline{B}\right) = q_e , \tag{61}$$

$$\nabla \cdot \underline{B} = 0 , \tag{62}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} , \tag{63}$$

$$\nabla \times \frac{\underline{B}}{\mu} + \nabla \times (\varepsilon_p \underline{v} \times \underline{E}) = \frac{\partial}{\partial t}\left(\varepsilon \underline{E} + \varepsilon_p \underline{v} \times \underline{B}\right) + q_e \underline{v} + \sigma_1 \underline{\hat{E}} + \sigma_2 \underline{\underline{d}} \cdot \underline{\hat{E}} + \sigma_4 \nabla T + \sigma_5 \underline{\underline{d}} \cdot \nabla T + \sigma_7 \underline{\hat{E}} \times \underline{B} + T^{-1} \kappa_{10} \nabla T \times \underline{B} . \tag{64}$$

Conservation of mass:

$$\nabla \cdot \underline{v} = 0 . \tag{65}$$

Conservation of linear momentum:

$$\begin{aligned} \rho \frac{D\underline{v}}{Dt} = & -\rho g[1 - \alpha(T - T_0)]\underline{i}_3 \; - \nabla\left(p + p_e + p_m\right) \\ & + \nabla \cdot \left(\mu_v (\nabla \underline{v} + \nabla \underline{v}^t)\right) - \nabla \cdot \left(\sigma_2 (\underline{\hat{E}} \otimes \underline{\hat{E}})\right) \\ & - \nabla \cdot \left(\frac{\kappa_2}{T}(\nabla T \otimes \nabla T)\right) - \nabla \cdot \left(\left(\frac{\kappa_5}{T} + \sigma_5\right)(\underline{\hat{E}} \otimes \nabla T)_s\right) \\ & + q_e \underline{\hat{E}} \; + \sigma_1 \underline{\hat{E}} \times \underline{B} \; + \sigma_2 \underline{\underline{d}} \cdot \underline{\hat{E}} \times \underline{B} \; + \sigma_4 \nabla T \times \underline{B} \\ & + \sigma_5 \underline{\underline{d}} \cdot \nabla T \times \underline{B} + \sigma_7 (\underline{\hat{E}} \times \underline{B}) \times \underline{B} + \frac{\kappa_{10}}{T} (\nabla T \times \underline{B}) \times \underline{B} \\ & + \varepsilon_p \left(\underline{\hat{E}} \cdot \nabla\right)\underline{E} + \frac{1}{\mu_m}\left(\underline{B} \cdot \nabla\right)\underline{B} - \varepsilon_p \left(\left(\underline{v} \times \underline{E}\right) \cdot \nabla\right)\underline{B} \\ & + \frac{D}{Dt}\left(\varepsilon_p (\underline{\hat{E}} \times \underline{B})\right). \end{aligned} \tag{66}$$

Conservation of energy:

$$\begin{aligned} \rho C_p \frac{DT}{Dt} = & \; Q_h + \nabla \cdot \left(\kappa_1 \nabla T + \kappa_2 \underline{\underline{d}} \cdot \nabla T + \kappa_4 \underline{\hat{E}}\right) \\ & + \nabla \cdot \left(\kappa_5 \underline{\underline{d}} \cdot \underline{\hat{E}} + \kappa_7 \nabla T \times \underline{B} + \kappa_{10} \underline{\hat{E}} \cdot \underline{B}\right) \\ & + \sigma_1 \underline{\hat{E}} \cdot \underline{\hat{E}} + \sigma_4 \underline{\hat{E}} \cdot \nabla T - \frac{\kappa_2}{T} \nabla T \cdot \underline{\underline{d}} \cdot \nabla T \\ & - \frac{\kappa_5}{T} \underline{\hat{E}} \cdot \underline{\underline{d}} \cdot \nabla T + \frac{\kappa_{10}}{T} \underline{\hat{E}} \cdot \left(\nabla T \times \underline{B}\right) \\ & + \underline{\hat{E}} \cdot \frac{D\left(\varepsilon_p \underline{\hat{E}}\right)}{Dt} - \left(\frac{\underline{B}}{\mu_m} - \varepsilon_p \underline{V} \times \underline{E}\right) \cdot \frac{D\underline{B}}{Dt} . \end{aligned} \tag{67}$$

The equation of electric charge conservation is omitted since it can be readily obtained by combining the first and fourth of Maxwell's equations. Here, the coefficient of viscosity

$$\mu_v = \alpha_1 / 2 \tag{68}$$

is used for the convenience of presentation. The viscous dissipation term and the unsteady pressure term on the right side of the energy equation have been neglected, as is usually done in incompressible viscous flow modeling (White, 1974).

NON-DIMENSIONAL PARAMETERS

Because of the extreme complexity of the complete, nonlinear, fully coupled EMHD model, it is practically impossible at the present time to contemplate development of a numerical simulation package for its integration. Consequently, the complete general EMHD model should be simplified for particular circumstances. This simplification must be performed in a consistent manner starting from the general EMHD model. Elimination of certain terms could be justified by an order of magnitude analysis that is most efficiently accomplished by performing a complete non-dimensionalization of the model.

Thus all flow-field, electric field, and magnetic field parameters and all physical properties will be non-dimensionalized. The reference length and velocity are denoted by L and V. The reference time will be defined by the ratio, L/V. The reference values of the pressure, electric field, and magnetic flux are taken to be ρV^2, E_0, and B_0, respectively. Temperature, however, is nondimensionalized by the equation

$$T = T_0 + \Delta T \cdot \theta . \tag{69}$$

Additional reference values of each parameter will be designated with a subscript $*$.

A very important parameter that will help in the elimination of a number of terms in the EMHD model is the squared ratio of the fluid speed and the speed of light, that is, square of electromagnetic Mach number.

$$\beta = \varepsilon_0 \mu_0 V^2 = \frac{V^2}{c^2} << 1 . \tag{70}$$

For example, $\beta \approx 10^{-14}$ when $V = 20$ m/s, which is the typical upper bound for the speed of an aqueous solution at room temperature and atmospheric pressure, in order to avoid cavitation.

Other non-dimensional parameters can be defined as:

$$\bar{\varepsilon}_p = \frac{\varepsilon_{p*}}{\varepsilon_*}, \quad \varepsilon_{r*} = \frac{\varepsilon_*}{\varepsilon_0}, \quad \mu_{r*} = \frac{\mu_*}{\mu_0}, \tag{71}$$

$$N_B = \frac{V B_0}{E_0}, \quad N_q = \frac{q_{e*} L}{\varepsilon_* E_0}, \quad N_T = \frac{\Delta T}{T_0}, \tag{72}$$

$$N_{\sigma 1} = \sigma_{1*} \mu_* V L = \text{magnetic Reynolds number} , \tag{73}$$

$$N_{\sigma 2} = \sigma_{2*} \mu_* V^2 , \quad N_{\sigma 4} = \frac{\sigma_{4*} \mu_* \Delta T V}{\quad}, \tag{74}$$

$$K_{\sigma 1} = \frac{\sigma_{1*} E_0 B_0 L}{\rho V^2}, \quad K_{\sigma 4} = \frac{\sigma_{4*} B_0 \Delta T}{\rho V^2}, \tag{82}$$

$$K_{\sigma 7} = \frac{\sigma_{7*} E_0 B_0^2}{\rho V^2}, \quad K_{\kappa 10} = \frac{\kappa_{10*} B_0^2}{\rho V^2}, \tag{83}$$

$$N_{es} = \frac{\varepsilon_{p*} E_0^2}{\rho V^2}, \quad N_{ms} = \frac{B_0^2}{\rho V^2 \mu_{m*}}, \tag{84}$$

$$Pe = \frac{\rho C_{p*} V L}{\kappa_{1*}} = \text{Peclet number} , \tag{85}$$

$$N_{\kappa 2} = \frac{\kappa_{2*}}{\rho C_{p*} L^2}, \quad N_{\kappa 4} = \frac{\kappa_{4*} E_0}{\rho C_{p*} \Delta T V}, \tag{86}$$

$$N_{\kappa 5} = \frac{\kappa_{5*} E_0}{\rho C_{p*} \Delta T L}, \quad N_{\kappa 7} = \frac{\kappa_{7*} B_0}{\rho C_{p*} V L}, \tag{87}$$

$$N_{\kappa 10} = \frac{\kappa_{10*} E_0 B_0}{\rho C_{p*} \Delta T V}, \quad N_{Q_h} = \frac{Q_{h*} L}{\rho C_{p*} \Delta T V}, \tag{88}$$

$$N_{\sigma 5} = \frac{\sigma_{5*}\mu_* \Delta T\, V^2}{E_0\, L}, \quad N_{\sigma 7} = \sigma_{7*}\mu_* B_0 V\, L\,, \tag{75}$$

$$\tilde{N}_{\kappa 10} = \frac{\kappa_{10*}\mu_* V B_0}{E_0}, \tag{76}$$

$$Re = \frac{\rho V L}{\mu_v} = \text{Reynolds number}\,, \tag{77}$$

$$K_{\sigma 2} = \frac{\sigma_{2*} E_0^2}{\rho V^2}, \quad K_{\kappa 2} = \frac{\kappa_{2*}\,\Delta T}{\rho V^2 L^2}, \tag{78}$$

$$K_{\kappa 5} = \frac{\kappa_{5*} E_0}{\rho V^2 L}, \quad K_{\sigma 5} = \frac{\sigma_{5*} E_0 \Delta T}{\rho V^2 L}, \tag{79}$$

$$Fr = \frac{V}{\sqrt{gL}} = \text{Froude number}\,, \tag{80}$$

$$N_{co} = \frac{q_{e*}\, E_0 L}{\rho V^2} = \text{Coulomb number}\,, \tag{81}$$

$$\Lambda_{\sigma 1} = \frac{\sigma_{1*} E_0^2 L}{\rho\, C_{p*} \Delta T\, V}, \quad \Lambda_{\sigma 4} = \frac{\sigma_{4*} E_0}{\rho\, C_{p*} V}, \tag{89}$$

$$Ec = \frac{V^2}{C_{p*}\Delta T} = \text{Eckert number}\,. \tag{90}$$

NON-DIMENSIONAL GOVERNING EQUATIONS FOR NON-LINEAR EMHD MODEL

Non-dimensional forms of the governing equations for the complete EMHD model will be presented below. No special notation is used for the non-dimensional field variables and all non-dimensional parameters are defined in the previous section. There is no restriction to the temporal and spatial variations of the material properties, such as conduction coefficients, permittivity, coefficient of viscosity, etc.

The non-dimensional EMHD Maxwell's equations are then:

$$\nabla \cdot \left(\varepsilon \underline{E}\right) + \bar{\varepsilon}_p N_B \nabla \cdot \left(\varepsilon_p \left(\underline{v} \times \underline{B}\right)\right) = N_q q_e\,, \tag{91}$$

$$\nabla \cdot \underline{B} = 0\,, \tag{92}$$

$$\nabla \times \underline{E} = -N_B \frac{\partial \underline{B}}{\partial t}, \tag{93}$$

$$
\begin{aligned}
&N_B \nabla \times \frac{\underline{B}}{\mu} + \beta\mu_{r*}\chi_{e*} \nabla \times \left(\varepsilon_p \left(\underline{v} \times \underline{E}\right)\right) = \\
&\beta\mu_{r*}\varepsilon_{r*}\left[\frac{\partial}{\partial t}\left(\varepsilon \underline{E}\right) + N_B \bar{\varepsilon}_p \frac{\partial}{\partial t}\left(\varepsilon_p \left(\underline{v} \times \underline{B}\right)\right) + N_q \left(q_e \underline{v}\right)\right] \\
&+ N_{\sigma 1}(\sigma_1 \underline{e}) + N_{\sigma 2}(\sigma_2 \underline{\underline{d}} \cdot \underline{e}) + N_{\sigma 4}(\sigma_4 \nabla\theta) \\
&+ N_{\sigma 5}(\sigma_5 \underline{\underline{d}} \cdot \nabla\theta) + N_{\sigma 7}(\sigma_7 \underline{e} \times \underline{B}) \\
&+ N_T \tilde{N}_{\kappa 10} \kappa_{10} \Theta^{-1} (\nabla\theta \times \underline{B}) \, .
\end{aligned}
\tag{94}
$$

Mass conservation equation:

$$
\nabla \cdot \underline{v} = 0 \, . \tag{95}
$$

Linear momentum equation:

$$
\begin{aligned}
\frac{D\underline{v}}{Dt} = &-\nabla p - N_{es} \nabla\left(\varepsilon_p \underline{e} \cdot \underline{e}/2\right) - N_{ms} \nabla\left(\mu_m^{-1} \underline{B} \cdot \underline{B}/2\right) \\
&- Fr^{-2}\left(1 - \alpha \Delta T \theta\right)\underline{i_3} + \frac{1}{Re} \nabla \cdot \left[\mu_v \left(\nabla \underline{v} + (\nabla \underline{v})^t\right)\right] \\
&- K_{\sigma 2} \nabla \cdot \left(\sigma_2 \underline{e} \otimes \underline{e}\right) - K_{\kappa 2} N_T \nabla \cdot \left(\Theta^{-1} \kappa_2 \nabla\theta \otimes \nabla\theta\right) \\
&- \nabla \cdot \left(\left(K_{\kappa 5} N_T \Theta^{-1} \kappa_5 + K_{\sigma 5} \sigma_5\right)\left(\underline{e} \otimes \nabla\theta\right)_s\right) \\
&+ N_{co}(q_e \underline{e}) + K_{\sigma 1}(\sigma_1 \underline{e} \times \underline{B}) + N_B K_{\sigma 2}\left[\sigma_2 (\underline{\underline{d}} \cdot \underline{e}) \times \underline{B}\right] \\
&+ K_{\sigma 4}(\sigma_4 \nabla\theta \times \underline{B}) + N_B K_{\sigma 5}\left(\sigma_5 (\underline{\underline{d}} \cdot \nabla\theta) \times \underline{B}\right) \\
&+ K_{\sigma 7}\left(\sigma_7 (\underline{e} \times \underline{B}) \times \underline{B}\right) + N_{\kappa 10} N_T \Theta^{-1}\left(\kappa_{10} (\nabla\theta \times \underline{B}) \times \underline{B}\right) \\
&+ N_{es} \varepsilon_p \left(\underline{e} \cdot \nabla\right)\underline{E} \;\; + N_{ms} \frac{1}{\mu_m}\left(\underline{B} \cdot \nabla\right)\underline{B} \\
&- N_B N_{es} \varepsilon_p \left(\left(\underline{v} \times \underline{E}\right) \cdot \nabla\right)\underline{B} \;\; + N_B N_{es} \frac{D}{Dt}\left(\varepsilon_p (\underline{e} \times \underline{B})\right).
\end{aligned}
\tag{96}
$$

Energy conservation equation:

$$
\begin{aligned}
C_p \frac{DT}{Dt} = &N_{Q_h} Q_h + Pe^{-1} \nabla \cdot \left(\kappa_1 \nabla\theta\right) + N_{\kappa 2} \nabla \cdot \left(\kappa_2 \underline{\underline{d}} \cdot \nabla\theta\right) \\
&+ N_{\kappa 4} \nabla \cdot \left(\kappa_4 \underline{e}\right) + N_{\kappa 5} \nabla \cdot \left(\kappa_5 \underline{\underline{d}} \cdot \underline{e}\right) + N_{\kappa 7} \nabla \cdot \left(\kappa_7 \nabla\theta \times \underline{B}\right) \\
&+ N_{\kappa 10} \nabla \cdot \left(\kappa_{10} \underline{e} \times \underline{B}\right) + \Lambda_{\sigma 1}(\sigma_1 \underline{e} \cdot \underline{e}) \\
&+ \Lambda_{\sigma 4}(\sigma_4 \underline{e} \cdot \nabla\theta) - N_{\kappa 2} N_T \Theta^{-1} \kappa 2 \nabla\theta \cdot \underline{\underline{d}} \cdot \nabla\theta \\
&- N_{\kappa 5} N_T \Theta^{-1} \kappa_5 \underline{e} \cdot \underline{\underline{d}} \cdot \nabla\theta + N_{\kappa 10} N_T \Theta^{-1} \kappa_{10} \underline{e} \cdot \left(\nabla\theta \times \underline{B}\right) \\
&+ Ec \, N_{es} \underline{e} \cdot \frac{D(\varepsilon_p \underline{e})}{Dt} - Ec\left(N_{ms} \frac{\underline{B}}{\mu_m} - N_B N_{es} \varepsilon_p \underline{v} \times \underline{E}\right) \cdot \frac{D\underline{B}}{Dt} \, .
\end{aligned}
\tag{97}
$$

Here, two additional quantities were introduced for compactness,

$$
\underline{e} = \underline{E} + N_B \underline{v} \times \underline{B}, \qquad \Theta = 1 + N_T \theta \, . \tag{98}
$$

Note, however, that the electro-magnetic field contains only three unknowns: $\underline{E}$, $\underline{B}$, and q_e.

CONCLUSIONS

Starting from very general fundamental principles and material constitutive relations, a complete analytical model (EMGD) was outlined for combined influence of unsteady electric and magnetic fields on a moving fluid that is polarizable and magnetizable. A simpler model of the EMGD was then derived by neglecting the terms which are higher than the second order in the original complete EMGD model. Finally, a complete analytical model for the incompressible flow-field under the combined influence of unsteady electric and magnetic fields (EMHD) was derived in a consistent manner. The EMHD model allows for non-linear, cross effects of electro-magnetic field and clearly specifies which physical properties need to be known for the successful modeling of such flows. The EMHD model was also represented in its non-dimensional fully consistent form.

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