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BOUNDARY CONDITIONS FOR ELECTRO-MAGNETO-HYDRODYNAMICS

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ABSTRACT

A general mathematical model of characteristic and non-reflecting boundary conditions for a system of partial differential equations governing unified Electro-Magneto-Hydrodynamic (EMHD) flows is developed. The novelty of this work is that it includes effects of linear polarization and magnetization of the media. A formulation using characteristic wave relations at the boundaries is derived for the extended Maxwell's equations and the extended Navier-Stokes equations so that it can be used to specify inflow and outflow free boundary conditions. The emphasis is on using recent theoretical results on well-posedness of boundary conditions for standard Navier-Stokes and Maxwell's equations to construct a systematic method for specifying the physical and numerical boundary conditions for electro-magneto-hydrodynamics. The formulation is fully three-dimensional and suitable for direct numerical implementation. The results show strong influence of electric and magnetic fields on fluid flow characteristics.

NOMENCLATURE

$\underline{\mathbf{B}}$ = magnetic flux density vector, kg A^{-1} s^{-2}
$\underline{\mathbf{D}} = \varepsilon_o \underline{\mathbf{E}} + \underline{\mathbf{P}}$ = electric displacement field vector, A s m^{-2}
e = internal energy per unit mass, m^2 s^{-2}
e_Σ = energy source on discontinuity surface, m^2 s^{-2}
$\underline{\mathbf{E}}$ = electric field vector, kg m s^{-3} A^{-1}, or V m^{-1}
$\underline{\mathsf{E}} = \underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}$ = electromotive intensity, kg m s^{-3} A^{-1}

$\underline{\mathbf{f}}$ = mechanical body force vector per unit mass, m s^{-2}
h = heat source or sink per unit mass, m^2 s^{-3}
h_Σ = heat source or sink on a line singularity, m^2 s^{-3}
$\underline{\mathbf{H}} = \underline{\mathbf{B}} / \mu_o - \underline{\mathbf{M}}$ = magnetic field intensity vector, A m^{-1}
$\underline{\mathbf{J}} = \underline{\mathbf{J}}_c + \underline{\mathbf{J}}_d$ = electric current density vector, A m^{-2}
$\underline{\mathbf{J}}_c$ = electric conduction current vector, A m^{-2}
$\underline{\mathbf{J}}_d = q_o \underline{\mathbf{v}}$ = electric drift current vector, A m^{-2}
$\underline{\mathbf{M}}$ = total magnetization vector per unit volume, A m^{-1}
$\underline{\mathsf{M}} = \underline{\mathbf{M}} + \underline{\mathbf{v}} \times \underline{\mathbf{P}}$ = magnetomotive intensity vector per unit volume, A m^{-1}
$\hat{\mathbf{n}}$ = outward unit normal vector
$\hat{\mathbf{n}}_\Gamma$ = outward unit normal vector to surface Γ
p = pressure, kg m^{-1} s^{-2}
$\underline{\mathbf{P}} = \varepsilon_P \underline{\mathbf{E}}$ = total polarization vector per unit volume, A s m^{-2}
q_s = surface free electric charge per unit volume, A s m^{-3}
q_o = total or free electric charge per unit volume, A s m^{-3}
$\underline{\dot{\mathbf{q}}}$ = heat flux vector, kg s^{-3}
$\underline{\dot{\mathbf{q}}}_\Lambda$ = heat flux vector on a line singularity Λ, kg s^{-3}
S = general surface
$\underline{\underline{\mathbf{t}}} = \underline{\underline{\mathbf{I}}} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$ = tangential projection matrix onto S
T = absolute temperature, K
$\underline{\mathbf{v}}$ = fluid velocity vector, m s^{-1}
$\underline{\mathbf{v}}_\Sigma$ = surface discontinuity velocity, m s^{-1}

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Greek letters

α_{1-18} = constitutive coefficients in electro-magnetic stress

β = Chorin's artificial compressibility coefficient

ε = dielectric constant (electric permittivity), $kg^{-1}\ m^{-3}\ s^4 A^2$

$\varepsilon_o = 8.854 \times 10^{-12}$ = vacuum dielectric constant or electric permittivity, $kg^{-1}\ m^{-3}\ s^4\ A^2$

κ_{1-12} = constitutive coefficients in heat conduction

μ = magnetic permeability coefficient, $kg\ m\ A^{-2}\ s^{-2}$

$\mu_o = 4\pi \times 10^{-7}$ =magnetic permeability of vacuum, $kg\ mA^2 s^{-2}$

$\chi^E = \varepsilon / \varepsilon_o - 1$ = electric susceptibility

$\chi^M = \mu / \mu_o - 1$ = magnetic susceptibility

ρ = fluid density, $kg\ m^{-3}$

σ_{1-12} = constitutive coefficients in conduction current

$\underline{\underline{\tau}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}^v + \underline{\underline{\tau}}^{EM}$ = combined stress tensor, $kg\ m^{-1}\ s^{-2}$

$\underline{\underline{\tau}}^v = 2\mu_v \underline{\underline{d}}$ = viscous stress tensor, $kg\ m^{-1}\ s^{-2}$

$\underline{\underline{\tau}}^{EM}$ = electromagnetic stress tensor, $kg\ m^{-1}\ s^{-2}$

Γ = arbitrary surface

Σ = discontinuity surface

Λ = discontinuity line

$\hat{\pi}$ = surface polarization density

Superscripts

EM = electric and magnetic related effects

v = viscous effects

Subscripts

EM = electric and magnetic related effects

v = viscous effects

o = reference value

INTRODUCTION

We will use the expression Electro-Magneto-Fluid Dynamics (EMFD) to describe phenomena when both magnetic and electric fields are simultaneously applied to an electrically conducting fluid containing electrically charged particles. Recently, a series of rigorous theoretical continuum mechanics treatments of unified EMFD flows (Eringen and Maugin, 1990a, 1990b; Dulikravich and Lynn, 1995a, 1995b, 1997a, 1997b; Dulikravich, 1998) have been developed. Due to the extreme complexity of the coupled system of Navier-Stokes, Maxwell's and constitutive equations describing these combined EMFD flows, the problem has classically been divided into two separate, simplified categories (Stuetzer, 1962): electro-hydrodynamics (EHD) studying flows under the influence of an electric field with free electric charges and no magnetic field (Dulikravich et al., 1993a; 1994a), and magneto-hydrodynamics (MHD) studying flows under the influence of a magnetic field and no free electric charges or electric fields (Lee and Dulikravich, 1991; Dulikravich et al., 1993b; 1994b).

The unified EMFD model reveals a host of inconsistencies and shortcomings of classical EHD and MHD formulations and allows discussion of the relative importance of terms describing the electro-magnetic force, electric current and heat transfer.

The ultimate objective of the present effort is to establish an analytical foundation that will apply directly to the numerical simulation capability of electromagnetic phenomena involved in fluid dynamics. The process involves the transition of the analysis techniques from computational fluid dynamics (CFD) to the computational electro-magneto-fluid dynamics (CEMFD) for flow analysis and control. A mathematical model of EMFD that is amenable to the numerical implementations has already been developed (Dulikravich and Jing, 1996; Dulikravich, 1998).

In the present paper, the objective is to reconstruct a fully conservative form of the EMFD governing equations and to specify procedures to define open boundary conditions. The EMFD system, without any account of medium polarization and magnetization, has already been analyzed and numerically simulated by Shankar et al. (1989) and Shang (1991).

The most important contribution of this paper is its inclusion of some important effects of polarization and magnetization in the boundary condition formulations.

The boundary conditions to be derived in this paper for the Navier-Stokes subsystem of the unified EMFD system will reduce to boundary conditions for the Euler system of inviscid fluid mechanics when the viscous, electric, and magnetic terms are omitted. The boundary conditions to be derived for Maxwell's subsystem of the unified EMHD system will reduce to Shang's (1991) characteristic boundary conditions when polarization and magnetization terms are omitted.

The method presented here is an extension of Thompson's (1987; 1990) type non-reflecting boundary conditions for hyperbolic equations. This formulation has the ability to accurately control different waves which cross the boundaries. It does not use any extrapolation procedure so that any arbitrariness in the formulation of the boundary conditions is avoided. The number of boundary conditions to be specified for the Navier-Stokes equations and Maxwell's equations will be obtained by theoretical analysis of well-posedness (Strikwerda, 1977; Dutt, 1988).

The method to be presented here for specifying boundary conditions for Navier-Stokes equations is certainly not the only one available. Several other procedures may be found in the literature (Dutt, 1988; Grinstein et al., 1987; Wanajakshi et al. 1989; Poinsot and Lele, 1992; Berenger, 1994). However, different limitations apply to each of these methods. For example, the procedure suggested by Poinsot and Lele is essentially an inviscid formulation supplemented with additional viscous conditions. It has one theoretical weakness in terms of lacking the rigorous indications on how to choose these viscous conditions, although the number and choices of physical boundary conditions (inviscid and viscous) may be guided by the theoretical studies of well-posedness (Strikwerda, 1977; Oliger and Sundstrom, 1978). Furthermore, the compatibility of the inviscid conditions with viscous conditions is not automatically ensured.

The limitation of method that we will use, suggested by the theoretical considerations, is the use of the characteristic concept that treats the Navier-Stokes equations for any Reynolds number. It utilizes an important assumption whose validity was supported by numerical investigations, despite its questionable mathematical justification (Dulikravich et al., 1994b; Poinsot and Lele, 1992).

This paper is organized into three parts. The first part will summarize the EMFD governing equations and provide details of the fully conservative form of the EMFD system. The second part will provide jump conditions to be used at solid boundaries. The third part will provide a detailed description of the entire procedure to derive and specify characteristic and non-reflecting boundary conditions to be used at free boundaries.

CONSERVATIVE FORMS OF EMHD SYSTEM

A complete set of generalized conservation laws governing EMFD has been derived by Eringen and Maugin (1990a) using a rigorous continuum mechanics approach. Dulikravich and Lynn (1995a; 1995b; 1997a; 1997b) have summarized these derivations and Dulikravich and Jing (1996) have formulated fully conservative forms of the EMHD system.

We will assume that fluid is Newtonian, incompressible, homocompositional, and that it has linear polarization and linear magnetization properties. Frequency of the applied electric and magnetic fields should be limited to less than approximately 1000 Hz and the fluid speed should be considerably lower than the speed of light for this mathematical model to be realistic. These are the only assumptions to be used in this model which will be referred to as a unified electro-magneto-hydrodynamics (EMHD).

It can be shown (Dulikravich and Jing, 1996) that a compact vector form of the unified EMHD system can be written as a combination of the Maxwell's sub-system (seven PDE's)

$$\frac{\partial \underline{\mathbf{E}}}{\partial t} - \nabla\times\left(\frac{\bar{\mu}}{\varepsilon_o}\underline{\mathbf{B}} + \chi^E \underline{\mathbf{v}}\times\left(\underline{\mathbf{E}} + \underline{\mathbf{v}}\times\underline{\mathbf{B}}\right)\right) = \underline{\mathbf{S}}^E \quad (1)$$

$$\frac{\partial \underline{\mathbf{B}}}{\partial t} + \nabla\times\underline{\mathbf{E}} = \underline{\mathbf{0}} \quad (2)$$

$$\frac{\partial q_o}{\partial t} + \nabla\cdot\underline{\mathbf{J}} = 0 \quad (3)$$

and the Navier-Stokes sub-system (five PDE's).

$$\nabla\cdot\underline{\mathbf{v}} = 0 \quad (4)$$

$$\frac{\partial \underline{\mathbf{v}}}{\partial t} + \nabla\cdot\left(\underline{\mathbf{v}}\underline{\mathbf{v}} - \frac{1}{\rho}\underline{\underline{\tau}}\right) - \frac{1}{\rho}\nabla\cdot\left(\underline{\mathbf{v}}\left(\underline{\mathbf{P}}\times\underline{\mathbf{B}}\right)\right) - \frac{1}{\rho}\nabla\cdot\left(\left(\underline{\mathbf{B}}\cdot\underline{\mathbf{M}}\right)\underline{\underline{\mathbf{I}}} + \left(\underline{\mathbf{E}}\cdot\underline{\mathbf{P}}\right)\underline{\underline{\mathbf{I}}}\right) = \underline{\mathbf{S}}^v \quad (5)$$

$$\frac{\partial e}{\partial t} + \nabla\cdot\left(e\underline{\mathbf{v}}\right) - \frac{1}{\rho}\left(\nabla\cdot\left(\underline{\underline{\tau}}\cdot\underline{\mathbf{v}}\right) - \nabla\cdot\underline{\dot{\mathbf{q}}}\right) = \underline{\mathbf{S}}^n \quad (6)$$

where the following dyadic identities were used in the linear momentum equation

$$\left(\nabla\underline{\mathbf{B}}\right)\cdot\underline{\mathbf{M}} = \nabla\cdot\left(\left(\underline{\mathbf{B}}\cdot\underline{\mathbf{M}}\right)\underline{\underline{\mathbf{I}}}\right) - \left(\nabla\underline{\mathbf{M}}\right)\cdot\underline{\mathbf{B}} \quad (7)$$

$$\left(\nabla\underline{\mathbf{E}}\right)\cdot\underline{\mathbf{P}} = \nabla\cdot\left(\left(\underline{\mathbf{E}}\cdot\underline{\mathbf{P}}\right)\underline{\underline{\mathbf{I}}}\right) - \left(\nabla\underline{\mathbf{P}}\right)\cdot\underline{\mathbf{E}} \quad (8)$$

Let us define the following terms as (Dulikravich and Jing, 1996)

$$\underline{\mathbf{R}} = \rho\underline{\mathbf{f}} + q_o\underline{\mathbf{E}} + \underline{\mathbf{J}}\times\underline{\mathbf{B}} + \left(\nabla\underline{\mathbf{E}}\right)\cdot\underline{\mathbf{P}} + \left(\nabla\underline{\mathbf{B}}\right)\cdot\underline{\mathbf{M}} + \nabla\cdot\left(\underline{\mathbf{v}}\left(\underline{\mathbf{P}}\times\underline{\mathbf{B}}\right)\right) - \nabla\cdot\left(\underline{\mathbf{v}}\rho\underline{\mathbf{v}} - \underline{\underline{\tau}}\right) \quad (9)$$

$$\underline{\mathbf{D}}_t = \nabla\times\left(\frac{\underline{\mathbf{B}}}{\mu_o} - \underline{\mathbf{M}}\right) - \underline{\mathbf{J}} \quad (10)$$

$$\bar{\mu} = \frac{1}{\mu_o\left(1+\chi^M\right)} = \frac{1}{\mu} \quad (11)$$

$$\bar{\varepsilon} = \frac{1}{\varepsilon_o\left(1+\chi^E\right)} = \frac{1}{\varepsilon} \quad (12)$$

$$\varepsilon_P = \varepsilon_o\chi^E = \varepsilon - \varepsilon_o \quad (13)$$

$$A = \frac{\varepsilon_P}{\rho\left(1+\chi^E\right) + \varepsilon_P\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}} \quad (14)$$

Then, the source terms can be given in a compact vector form as (Dulikravich and Jing, 1996)

$$\underline{\mathbf{S}}^{E}=\frac{1}{\varepsilon_o}\Big\{-\underline{\mathbf{J}}-\rho A/\varepsilon_o\underline{\mathbf{D}}_t-\rho A(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}$$
$$+A\big(\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})-\underline{\mathbf{R}}\big)\times\mathbf{B} \tag{15}$$
$$-\bar{\varepsilon}\chi^{E}A\big(\big(\underline{\mathbf{D}}_t+\varepsilon_o(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\big)\cdot\underline{\mathbf{B}}\big)\underline{\mathbf{B}}\Big\}$$

$$\underline{\mathbf{S}}^{v}=\underline{\mathbf{f}}+\frac{1}{\rho}\Big\{q_o\underline{\mathbf{E}}+\rho A\big((\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\big)\times\underline{\mathbf{B}}$$
$$+A\big(\big(\underline{\mathbf{R}}-\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\big)\times\underline{\mathbf{B}}\big)\times\underline{\mathbf{B}}$$
$$+\rho A/\varepsilon_o\underline{\mathbf{D}}_t\times\underline{\mathbf{B}}+(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{P}}+\underline{\mathbf{J}}\times\underline{\mathbf{B}}\Big\} \tag{16}$$
$$-\frac{1}{\rho}\big((\nabla\underline{\mathbf{M}})\cdot\underline{\mathbf{B}}+(\nabla\underline{\mathbf{P}})\cdot\underline{\mathbf{E}}\big)$$

$$\underline{\mathbf{S}}^{n}=h+\frac{1}{\rho}(\underline{\mathbf{E}}+\underline{\mathbf{v}}\times\underline{\mathbf{B}})\cdot\big[(\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{P}}$$
$$+A\big(\underline{\mathbf{R}}-\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\big)\times\underline{\mathbf{B}}$$
$$+\bar{\varepsilon}\varepsilon_P A\big(\big(\underline{\mathbf{D}}_t+\varepsilon_o(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\big)\cdot\underline{\mathbf{B}}\big)\underline{\mathbf{B}} \tag{17}$$
$$+\frac{\rho A}{\varepsilon_o}\underline{\mathbf{D}}_t+A(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\big]$$
$$+\frac{1}{\rho}\underline{\mathbf{J}}_c\cdot(\underline{\mathbf{E}}+\underline{\mathbf{v}}\times\underline{\mathbf{B}})$$
$$-\frac{1}{\rho}\bar{\mu}\chi^{M}\ \underline{\mathbf{B}}\cdot\ \big((\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{B}}-\nabla\times\underline{\mathbf{E}}\big)$$

Notice that these source terms are formulated in such a way as not to have explicit time derivatives (Dulikravich and Jing, 1996).

The electro-magnetic stress tensor for a non-linear fluid is given as (Eringen and Maugin, 1990a, pp.177-178)

$$\underline{\underline{\tau}}^{EM}=\alpha_1\underline{\underline{\mathbf{d}}}^2+\alpha_2\underline{\mathbf{E}}\otimes\underline{\mathbf{E}}+\alpha_3\underline{\mathbf{B}}\otimes\underline{\mathbf{B}}$$
$$+\alpha_4\nabla T\otimes\nabla T+\alpha_5\big(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}}\big)_S$$
$$+\alpha_6\big(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}}\big)_S+\alpha_7\big(\nabla T\otimes\underline{\underline{\mathbf{d}}}\cdot\nabla T\big)_S$$
$$+\alpha_8\big(\nabla T\otimes\underline{\underline{\mathbf{d}}}^2\cdot\nabla T\big)_S+\alpha_9\big(\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}-\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\big)$$
$$+\alpha_{10}\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}+\alpha_{11}\big(\underline{\underline{\mathbf{d}}}^2\cdot\underline{\underline{\mathbf{W}}}-\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}^2\big)$$
$$+\alpha_{12}\big(\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}^2-\underline{\underline{\mathbf{W}}}^2\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}\big) \tag{18}$$
$$+\alpha_{13}\big(\underline{\mathbf{E}}\otimes\nabla T\big)_S+\alpha_{14}\big(\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\otimes\underline{\mathbf{E}}\cdot\underline{\underline{\mathbf{W}}}\big)_S$$
$$+\alpha_{15}\big(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\big)_S+\alpha_{16}\big(\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{W}}}^2\cdot\underline{\mathbf{E}}\big)_S$$
$$+\alpha_{17}\big(\underline{\underline{\mathbf{W}}}\cdot(\underline{\mathbf{E}}\otimes\nabla T-\nabla T\otimes\underline{\mathbf{E}})\big)_S$$
$$+\alpha_{18}\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\otimes\nabla T-\nabla T\otimes\underline{\mathbf{E}})$$
$$-\alpha_{18}(\underline{\mathbf{E}}\otimes\nabla T-\nabla T\otimes\underline{\mathbf{E}})\cdot\underline{\underline{\mathbf{d}}}$$

where $\underline{\underline{\mathbf{W}}}=W_{ij}=\varepsilon_{ijk}B_k$, while the subscript s indicates symmetrization. The unified EMHD electric conduction current in a non-linear media can be written as (Eringen and Maugin, 1996, pp.162).

$$\underline{\mathbf{J}}_c=\sigma_1\underline{\mathbf{E}}+\sigma_2\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}}+\sigma_3\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}}+\sigma_4\nabla T$$
$$+\sigma_5\underline{\underline{\mathbf{d}}}\cdot\nabla T+\sigma_6\underline{\underline{\mathbf{d}}}^2\cdot\nabla T+\sigma_7\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$
$$+\sigma_8\big(\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})-(\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}})\times\underline{\mathbf{B}}\big)+\sigma_9\nabla T\times\underline{\mathbf{B}} \tag{19}$$
$$+\sigma_{10}\big(\underline{\underline{\mathbf{d}}}\cdot(\nabla T\times\underline{\mathbf{B}})-(\underline{\underline{\mathbf{d}}}\cdot\nabla T)\times\underline{\mathbf{B}}\big)$$
$$+\sigma_{11}(\underline{\mathbf{B}}\cdot\underline{\mathbf{E}})\underline{\mathbf{B}}+\sigma_{12}(\underline{\mathbf{B}}\cdot\nabla T)\underline{\mathbf{B}}$$

Similarly, the constitutive relation for heat flux is given as (Eringen and Maugin, 1996, pp.161)

$$\dot{\underline{\mathbf{q}}}=\kappa_1\underline{\mathbf{E}}+\kappa_2\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}}+\kappa_3\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}}+\kappa_4\nabla T$$
$$+\kappa_5\underline{\underline{\mathbf{d}}}\cdot\nabla T+\kappa_6\underline{\underline{\mathbf{d}}}^2\cdot\nabla T+\kappa_7\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$
$$+\kappa_8\big(\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})-(\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}})\times\underline{\mathbf{B}}\big)+\kappa_9\nabla T\times\underline{\mathbf{B}} \tag{20}$$
$$+\kappa_{10}\big(\underline{\underline{\mathbf{d}}}\cdot(\nabla T\times\underline{\mathbf{B}})-(\underline{\underline{\mathbf{d}}}\cdot\nabla T)\times\underline{\mathbf{B}}\big)$$
$$+\kappa_{11}(\underline{\mathbf{B}}\cdot\underline{\mathbf{E}})\underline{\mathbf{B}}+\kappa_{12}(\underline{\mathbf{B}}\cdot\nabla T)\underline{\mathbf{B}}$$

Here, α_i, σ_i and κ_i are the coefficients providing physical properties of the medium. At the present time, values of only a few of these coefficients are known even for the most commonly used fluids.

Expressions for total polarization and magnetization of a non-linear medium can be modeled with expressions of similar complexity (Eringen and Maugin, 1990a, pp.175).

If we limit the analysis to fluids with only linear polarization and linear magnetization, then we can use the following expressions

$$\underline{\mathbf{P}} = \varepsilon_o \chi^E (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \tag{21}$$

$$\underline{\mathbf{M}} = \frac{\chi^M}{\mu_o (1+\chi^M)} \underline{\mathbf{B}} = \underline{\mathbf{M}} + \underline{\mathbf{v}} \times \underline{\mathbf{P}} \tag{22}$$

$$\underline{\mathbf{J}}_c = \sigma_1 (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) + \sigma_4 \nabla T + \sigma_7 (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \times \underline{\mathbf{B}} + \sigma_9 \nabla T \times \underline{\mathbf{B}} + \sigma_{11} (\underline{\mathbf{B}} \cdot (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})) \underline{\mathbf{B}} + \sigma_{12} (\underline{\mathbf{B}} \cdot \nabla T) \underline{\mathbf{B}} \tag{23}$$

$$\underline{\dot{\mathbf{q}}} = \kappa_1 (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) + \kappa_4 \nabla T + \kappa_7 (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \times \underline{\mathbf{B}} + \kappa_9 \nabla T \times \underline{\mathbf{B}} + \kappa_{11} (\underline{\mathbf{B}} \cdot (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})) \underline{\mathbf{B}} + \kappa_{12} (\underline{\mathbf{B}} \cdot \nabla T) \underline{\mathbf{B}} \tag{24}$$

The fully conservative vector form of the EMHD equations (1-6) can be rewritten in Cartesian coordinates as (Dulikravich and Jing, 1996)

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial t} + \frac{\partial \tilde{\mathbf{E}}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}}{\partial y} + \frac{\partial \tilde{\mathbf{G}}}{\partial z} = \tilde{\mathbf{S}} \tag{25}$$

where the solution vector of unknown quantities is given as

$$\tilde{\mathbf{Q}} = \left\{ E_x, E_y, E_z, B_x, B_y, B_z, q_o, \frac{p}{\beta}, v_x, v_y, v_z, e \right\}^* \tag{26}$$

Here, asterisk symbol designates transpose of a vector while coefficient β is Chorin's (1967) artificial compressibility which is used to create the unsteady term in the mass conservation of an otherwise incompressible fluid. The vector of source terms is given as

$$\tilde{\mathbf{S}} = \left\{ S_x^E, S_y^E, S_z^E, 0,0,0,0, S_x^v, S_y^v, S_z^v, S^n \right\}^* \tag{27}$$

The flux vectors are defined as

$$\tilde{\mathbf{E}} = \begin{Bmatrix} 0 \\ H_z/\varepsilon_o \\ -H_y/\varepsilon_o \\ 0 \\ -E_z \\ E_y \\ J_x \\ v_x \\ v_x^2 + \frac{1}{\rho}\left(p - \tau_{xx}^v - \tau_{xx}^{EM} - N_{EP} - N_{BM} - v_x K_x\right) \\ v_x v_y - \frac{1}{\rho}\left(\tau_{xy}^v + \tau_{xy}^{EM} + v_y K_x\right) \\ v_x v_z - \frac{1}{\rho}\left(\tau_{xz}^v + \tau_{xz}^{EM} + v_z K_x\right) \\ e v_x + \frac{1}{\rho}\left(\dot{q}_x - I_x\right) \end{Bmatrix} \tag{28}$$

$$\tilde{\mathbf{F}} = \begin{Bmatrix} -H_z/\varepsilon_o \\ 0 \\ H_x/\varepsilon_o \\ E_z \\ 0 \\ -E_x \\ J_y \\ v_y \\ v_y v_x - \frac{1}{\rho}\left(\tau_{xy}^v + \tau_{xy}^{EM} + v_x K_y\right) \\ v_y^2 + \frac{1}{\rho}\left(p - \tau_{yy}^v - \tau_{yy}^{EM} - N_{EP} - N_{BM} - v_y K_y\right) \\ v_y v_z - \frac{1}{\rho}\left(\tau_{yz}^v + \tau_{yz}^{EM} + v_z K_y\right) \\ e v_y + \frac{1}{\rho}\left(\dot{q}_y - I_y\right) \end{Bmatrix} \tag{29}$$

$$\tilde{\mathbf{G}}=\begin{Bmatrix} H_y/\varepsilon_o \\ -H_x/\varepsilon_o \\ 0 \\ -E_y \\ E_x \\ 0 \\ J_z \\ v_z \\ v_z v_x - \frac{1}{\rho}\left(\tau^{v}_{xz}+\tau^{EM}_{xz}+v_x K_z\right) \\ v_z v_y - \frac{1}{\rho}\left(\tau^{v}_{yz}+\tau^{EM}_{yz}+v_y K_z\right) \\ v_z^2+\frac{1}{\rho}\left(p-\tau^{v}_{zz}-\tau^{EM}_{zz}-N_{EP}-N_{BM}-v_z K_z\right) \\ e v_z+\frac{1}{\rho}\left(\dot{q}_z-I_z\right) \end{Bmatrix} \tag{30}$$

Components of the magnetic field intensity vector, $\underline{\mathbf{H}}$, are

$$\begin{aligned} H_x &= \frac{B_x}{\mu} + v_y P_z - v_z P_y \\ H_y &= \frac{B_y}{\mu} + v_z P_x - v_x P_z \\ H_z &= \frac{B_z}{\mu} + v_x P_y - v_y P_x \end{aligned} \tag{31}$$

where components of the polarization vector, $\underline{\mathbf{P}}$, are

$$\begin{aligned} P_x &= \varepsilon_P(E_x + v_y B_z - v_z B_y) \\ P_y &= \varepsilon_P(E_y + v_z B_x - v_x B_z) \\ P_z &= \varepsilon_P(E_z + v_x B_y - v_y B_x) \end{aligned} \tag{32}$$

We can also write components of $\left(\underline{\mathbf{P}}\times\underline{\mathbf{B}}\right)$ as

$$\begin{aligned} K_x &= P_y B_z - P_z B_y \\ K_y &= P_z B_x - P_x B_z \\ K_z &= P_x B_y - P_y B_x \end{aligned} \tag{33}$$

In addition, we can define the terms

$$\begin{aligned} N_{BM} = {} & B_x(H_x - \frac{B_x}{\mu_o}) + B_y(H_y - \frac{B_y}{\mu_o}) \\ & + B_z(H_z - \frac{B_z}{\mu_o}) \end{aligned} \tag{34}$$

$$N_{EP} = E_x P_x + E_y P_y + E_z P_z \tag{35}$$

$$N_{BP} = B_x P_x + B_y P_y + B_z P_z \tag{36}$$

$$N_{BT} = B_x \frac{\partial T}{\partial x} + B_y \frac{\partial T}{\partial y} + B_z \frac{\partial T}{\partial z} \tag{37}$$

Components of the electric current vector, $\underline{\mathbf{J}}$, are

$$\begin{aligned} J_x = {} & v_x q_o + \frac{\sigma_1}{\varepsilon_P} P_x + \sigma_4 \frac{\partial T}{\partial x} + \frac{\sigma_7}{\varepsilon_P} K_x \\ & + \sigma_9(\frac{\partial T}{\partial y} B_z - \frac{\partial T}{\partial z} B_y) + \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_x + \sigma_{12} N_{BT} B_x \\[1em] J_y = {} & v_y q_o + \frac{\sigma_1}{\varepsilon_P} P_y + \sigma_4 \frac{\partial T}{\partial y} + \frac{\sigma_7}{\varepsilon_P} K_y \\ & + \sigma_9(\frac{\partial T}{\partial z} B_x - \frac{\partial T}{\partial x} B_z) + \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_y + \sigma_{12} N_{BT} B_y \\[1em] J_z = {} & v_z q_o + \frac{\sigma_1}{\varepsilon_P} P_z + \sigma_4 \frac{\partial T}{\partial z} + \frac{\sigma_7}{\varepsilon_P} K_z \\ & + \sigma_9(\frac{\partial T}{\partial x} B_y - \frac{\partial T}{\partial y} B_x) + \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_z + \sigma_{12} N_{BT} B_z \end{aligned} \tag{38}$$

$$\dot{q}_x = \frac{\kappa_1}{\varepsilon_P} P_x + \kappa_4 \frac{\partial T}{\partial x} + \frac{\kappa_7}{\varepsilon_P} K_x + \kappa_{12} N_{BT} B_x + \kappa_9 \left(\frac{\partial T}{\partial y} B_z - \frac{\partial T}{\partial z} B_y\right) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_x$$

$$\dot{q}_y = \frac{\kappa_1}{\varepsilon_P} P_y + \kappa_4 \frac{\partial T}{\partial y} + \frac{\kappa_7}{\varepsilon_P} K_y + \kappa_{12} N_{BT} B_y + \kappa_9 \left(\frac{\partial T}{\partial z} B_x - \frac{\partial T}{\partial x} B_z\right) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_y \quad (39)$$

$$\dot{q}_z = \frac{\kappa_1}{\varepsilon_P} P_z + \kappa_4 \frac{\partial T}{\partial z} + \frac{\kappa_7}{\varepsilon_P} K_z + \kappa_{12} N_{BT} B_z + \kappa_9 \left(\frac{\partial T}{\partial x} B_y - \frac{\partial T}{\partial y} B_x\right) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_z$$

$$I_x = v_x(-p + \tau^{V}_{xx} + \tau^{EM}_{xx}) + v_y(\tau^{V}_{xy} + \tau^{EM}_{xy}) + v_z(\tau^{V}_{xz} + \tau^{EM}_{xz})$$

$$I_y = v_x(\tau^{V}_{xy} + \tau^{EM}_{xy}) + v_y(-p + \tau^{V}_{yy} + \tau^{EM}_{yy}) + v_z(\tau^{V}_{yz} + \tau^{EM}_{yz}) \quad (40)$$

$$I_z = v_x(\tau^{V}_{xz} + \tau^{EM}_{xz}) + v_y(\tau^{V}_{yz} + \tau^{EM}_{yz}) + v_z(-p + \tau^{V}_{zz} + \tau^{EM}_{zz})$$

The most rigorous method for formulating boundary conditions for EMHD equations is to use the conservation equations themselves on the boundary to complement the set of physical boundary conditions. Variables which are not imposed by the boundary conditions are then computed on the boundaries by solving the same conservation equations as in the domain. Mathematically, the EMHD equations are characterized as an incompletely parabolic system of partial differential equations which consists of one parabolic subsystem coupled to a hyperbolic subsystem. For initial-boundary value problems that require the solution to remain bounded by the data at every time level, boundary conditions which make both subsystems well-posed render the global system well-posed, too (Strikwerda, 1977). This makes it possible to formulate boundary conditions for the Navier-Stokes and the Maxwell's equations separately (Lee and Dulikravich 1991; Dulikravich et al. 1994b).

JUMP CONDITIONS

The conservation laws on the discontinuity surfaces and discontinuity lines have been derived previously for the Navier-Stokes equations of fluid motion and Maxwell equations of electro-magnetics (Eringen and Maugin, 1990a; 1990b). They can be used to formulate interface boundary conditions for the EMHD governing equations. These so-called jump boundary conditions state the requirements for different field quantities at material (physical) or artificial (numerical) interfaces based on problem physics alone. Use of jump conditions for boundary conditions has become a *de facto* standard in electro-magnetics as a look at any electromagnetics fields textbook reveals (Johnk, 1988). However, fluid dynamics has seemingly bypassed the jump condition route to boundary condition formulations. The reason behind this disparity in the use of jump conditions perhaps stems from the fact that typically those using electro-magnetism must deal with the fact that the electro-magnetic fields often permeate partly or fully through the boundary. This condition is not found in the typical "solid wall" conditions of fluid mechanics, which are easily observed and understood physical characteristics. On the other hand, jump conditions are essentially the conservation laws on the discontinuity surfaces or the discontinuity lines. Hence, a jump condition route for formulating boundary conditions is essentially a conservation law route which is just slightly different from characteristic boundary conditions in terms of different starting forms of the same conservation laws.

Jump Conditions on Moving Discontinuities and on Solid Boundaries

As the conservation laws have already been established on the discontinuity surface Σ and line Λ (Eringen and Maugin, 1990a; 1990b), the jump conditions can be summarized as

$$\left[\rho\left(\underline{v} - \underline{v}_\Sigma\right)\right] \cdot \hat{\mathbf{n}}_r = 0 \qquad \text{on } \Sigma - \Lambda \quad (41)$$

$$\left[\rho \underline{v} \otimes \left(\underline{v} - \underline{v}_\Sigma\right) - \underline{\underline{\tau}}\right] \cdot \hat{\mathbf{n}}_r = 0 \qquad \text{on } \Sigma - \Lambda \quad (42)$$

$$\frac{\partial \rho e_\Sigma}{\partial t} + \left(\underline{v} \cdot \hat{\mathbf{n}}\right) \frac{\partial \rho e_\Sigma}{\partial n} + \left(\underline{\underline{t}} \cdot \nabla\right) \cdot \left(\rho e_\Sigma \underline{v}_\Sigma - \underline{\dot{q}}_\Lambda\right) + \left[\rho e\left(\underline{v} - \underline{v}_\Sigma\right) - \underline{\underline{\tau}} \cdot \underline{v} + \underline{\dot{q}}\right] \cdot \hat{\mathbf{n}}_r - \rho h_\Lambda = 0 \quad (43)$$

$$\left[\underline{\mathbf{D}}\right] \cdot \hat{\mathbf{n}}_r = q_s - \left(\underline{\underline{t}} \cdot \nabla\right) \cdot \tilde{\pi} \quad (44)$$

$$\left[\underline{\mathbf{B}}\right] \cdot \hat{\mathbf{n}}_r = 0 \quad (45)$$

$$\frac{\partial q_s}{\partial t}+(\underline{v}\cdot\hat{\mathbf{n}})\frac{\partial q_s}{\partial n}+(\underline{\underline{t}}\cdot\nabla)\cdot(q_s\mathbf{v}_\Sigma+\underline{\mathbf{J}}_s)$$
$$+\left[q_s(\underline{v}-\mathbf{v}_\Sigma)+\underline{\mathbf{J}}_c\right]\cdot\hat{\mathbf{n}}_\Gamma=0 \tag{46}$$

$$\left[\underline{\mathbf{B}}\times(\underline{v}-\underline{v}_\Sigma)+\underline{\mathbf{E}}+\underline{v}\times\underline{\mathbf{B}}\right]\cdot\hat{\mathbf{n}}_\Lambda=0 \tag{47}$$

$$\left[\underline{v}_\Sigma\times\underline{\mathbf{D}}-\underline{\mathbf{H}}\right]\times\hat{\mathbf{n}}_\Lambda=\underline{\mathbf{J}}_s \text{ on } \Sigma-\Lambda \tag{48}$$

If Σ is a material surface, that is, a surface whose geometrical points have an absolute velocity equal to that of the material particles occupying the same place instantaneously, then $\underline{v}=\underline{v}_\Sigma$. The jump conditions then reduce to those that are valid at the solid boundaries moving with the velocity $\underline{v}_\Sigma=\underline{v}$.

The boundary surface S of a material body is a particular case of the material surface. Hence, equations (41-48) are the relevant boundary conditions on $S-\Sigma$, with the surface distribution of a free charge, q_s, a surface current, $\underline{\mathbf{J}}_s$, and a surface polarization density, $\underline{\tilde{\pi}}$, prescribed on it. If these fields are supposedly zero, then equations (41-48) yield the following well-known boundary conditions

$$\left[\underline{\mathbf{D}}\right]\cdot\hat{\mathbf{n}}_\Gamma=0 \tag{49}$$

$$\left[\underline{\mathbf{B}}\right]\cdot\hat{\mathbf{n}}_\Gamma=0 \tag{50}$$

$$\left[\underline{\mathbf{E}}+\underline{v}\times\underline{\mathbf{B}}\right]\cdot\hat{\mathbf{n}}_\Lambda=0 \tag{51}$$

$$\left[\underline{v}\times\underline{\mathbf{D}}-\underline{\mathbf{H}}\right]\times\hat{\mathbf{n}}_\Lambda=0 \tag{52}$$

Various situations arising in quasi electro- or magneto-statics are easily described as particular cases of equations (41-52).

Solid wall boundary conditions for both Navier-Stokes equations and Maxwell's equations can be obtained from the jump conditions. The procedure to define solid wall boundary conditions could involve three steps. Let us consider a given solid wall boundary:

Step 1. For fluid and electromagnetic fields, construct jump conditions on the solid wall (material surface).

Step 2. Impose available physical boundary conditions and then eliminate the corresponding jump conditions which were obtained from step 1.

Step 3. Use the remaining jump conditions combined with the values of the specified variables obtained from step 2 to compute all variables which were not given by the physical boundary conditions implemented in step 2.

CHARACTERISTIC BOUNDARY CONDITIONS

For most boundary value problems of electro-magnetic dynamics, jump conditions are exclusively used to formulate solid wall boundary conditions where a discontinuity occurs. For the inflow and outflow boundaries where no surface or line discontinuities exist, this approach loses its physical basis and is theoretically invalid. Therefore, an alternative approach based on conservation laws for continuous surfaces or lines becomes necessary in order to formulate these boundary conditions. Among several possibilities, characteristic boundary condition formulations, which start from a non-conservative equivalent characteristic form of EMHD system instead of the original fully conservative form, are adopted in this work. This appealing approach has clear physical interpretations although, unfortunately, it still lacks rigorous mathematical proof.

Characteristic Boundary Conditions for Maxwell's Subsystem

Characteristic treatment of the Maxwell's subsystem can be performed by rewriting the conservative form of this sub-system of EMHD

$$\frac{\partial \mathbf{Q}_{EM}}{\partial t}+\frac{\partial \mathbf{E}_{EM}}{\partial x}+\frac{\partial \mathbf{F}_{EM}}{\partial y}+\frac{\partial \mathbf{G}_{EM}}{\partial z}=\mathbf{S}_{EM} \tag{53}$$

into its non-conservative (characteristic) form

$$\frac{\partial \mathbf{Q}_{EM}}{\partial t}+\underline{\underline{\mathbf{A}}}_{EM}\frac{\partial \mathbf{Q}_{EM}}{\partial x}$$
$$+\underline{\underline{\mathbf{B}}}_{EM}\frac{\partial \mathbf{Q}_{EM}}{\partial y}+\underline{\underline{\mathbf{C}}}_{EM}\frac{\partial \mathbf{Q}_{EM}}{\partial z}=\mathbf{S}_{EM} \tag{54}$$

where the electro-magnetic solution vector should be expressed in terms of the unknown primitive variables.

$$\mathbf{Q}_{EM}=\left\{E_x,E_y,E_z,B_x,B_y,B_z,q_o\right\}^* \tag{55}$$

The fluxes $\mathbf{E}_{EM},\mathbf{F}_{EM},\mathbf{G}_{EM}$ should also be expressed explicitly as the functions of the primitive variables.

This transformation could be performed by introducing the constitutive equations describing the material properties. For Newtonian fluids subject to applied electric and magnetic fields under the assumption of linear polarization and magnetization, we can use the equations (21-24).

For illustration purposes, the flux vector $\mathbf{E}_{EM}$ for the linear fluid is given as

$$\mathbf{E}_{\mathbf{EM}} = \begin{Bmatrix} 0 \\ H_z / \varepsilon_o \\ -H_y / \varepsilon_o \\ 0 \\ -E_z \\ E_y \\ J_x \end{Bmatrix} \tag{56}$$

The Jacobian matrix $\underline{\underline{\mathbf{A}}}_{\mathbf{EM}} = \partial \mathbf{E}_{\mathbf{EM}} / \partial \mathbf{Q}_{\mathbf{EM}}$ is then obtained as

$$\underline{\underline{\mathbf{A}}}_{\mathbf{EM}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} & a_{25} & a_{26} & 0 \\ a_{31} & 0 & a_{33} & a_{34} & a_{35} & a_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & v_x \end{bmatrix} \tag{57}$$

where the coefficients are

$$a_{21} = -\chi^E v_y \qquad a_{31} = -\chi^E v_z \tag{58}$$

$$a_{22} = \chi^E v_x \qquad a_{33} = \chi^E v_x \tag{59}$$

$$a_{24} = \chi^E v_x v_z \qquad a_{34} = -\chi^E v_x v_y \tag{60}$$

$$a_{25} = \chi^E v_y v_z \qquad a_{36} = -\chi^E v_y v_z \tag{61}$$

$$a_{26} = \frac{1}{\mu \varepsilon_o} - \chi^E \left(v_x^2 + v_y^2 \right) \tag{62}$$

$$a_{35} = -\frac{1}{\mu \varepsilon_o} + \chi^E \left(v_x^2 + v_z^2 \right) \tag{63}$$

$$a_{71} = \sigma_1 + \sigma_{11} B_x^2 \tag{64}$$

$$a_{72} = \sigma_7 B_z + \sigma_{11} B_x B_y \tag{65}$$

$$a_{73} = -\sigma_7 B_y + \sigma_{11} B_x B_z \tag{66}$$

$$a_{74} = \sigma_7 (v_y B_y + v_z B_z) + \sigma_{11} (E_x B_x + \frac{N_{BP}}{\varepsilon_P}) + \sigma_{12} (N_{BT} + B_x \frac{\partial T}{\partial x}) \tag{67}$$

$$a_{75} = -\sigma_1 v_z - \frac{\sigma_7}{\varepsilon_P} (P_z + \varepsilon_P v_x B_y) - \sigma_9 \frac{\partial T}{\partial z} + \sigma_{11} E_y B_x + \sigma_{12} B_x \frac{\partial T}{\partial y} \tag{68}$$

$$a_{76} = \sigma_1 v_y + \frac{\sigma_7}{\varepsilon_P} (P_y - \varepsilon_P v_x B_z) + \sigma_9 \frac{\partial T}{\partial y} + \sigma_{11} E_z B_x + \sigma_{12} B_x \frac{\partial T}{\partial z} \tag{69}$$

Jacobian matrices $\underline{\underline{\mathbf{B}}}_{\mathbf{EM}}$ and $\underline{\underline{\mathbf{C}}}_{\mathbf{EM}}$ in equation (51) may be obtained in the same fashion.

Eigenvalues of matrices larger than order four cannot be obtained with commercially available symbolic programming software. Consequently, tedious algebraic manipulations had to be performed by hand so that the vector of eigenvalues of the coefficient matrix $\underline{\underline{\mathbf{A}}}_{\mathbf{EM}}$ can be found as

$$\lambda_{\mathbf{EM}} = \left\{ 0, \lambda_E^+, \lambda_E^-, 0, \lambda_B^+, \lambda_B^-, v_x \right\}^* \tag{70}$$

That is, eigenvalues $\lambda_1 = \lambda_4 = 0$, $\lambda_7 = v_x$, while the remaining four eigenvalues can be obtained from the following fourth order algebraic equation

$$\lambda^4 + \alpha \lambda^3 + \nu \lambda^2 + \gamma \lambda + \delta = 0 \tag{71}$$

The four eigenvalues are the roots of this polynomial

$$\lambda_E^+ = -\frac{1}{4} \Phi_{\mathbf{EM1}} + \sqrt{\frac{1}{4} \Phi_{\mathbf{EM1}}^2 - \Omega_{\mathbf{EM1}}} \tag{72}$$

$$\lambda_E^- = -\frac{1}{4}\Phi_{EM1} - \sqrt{\frac{1}{4}\Phi_{EM1}^2 - \Omega_{EM1}} \quad (73)$$

$$\lambda_B^+ = -\frac{1}{4}\Phi_{EM2} + \sqrt{\frac{1}{4}\Phi_{EM2}^2 - \Omega_{EM2}} \quad (74)$$

$$\lambda_B^- = -\frac{1}{4}\Phi_{EM2} - \sqrt{\frac{1}{4}\Phi_{EM2}^2 - \Omega_{EM2}} \quad (75)$$

$$\Phi_{EM1} = \alpha_{EM} + \sqrt{\alpha_{EM}^2 - 4\nu_{EM} + 4\psi_{EM}} \quad (76)$$

$$\Phi_{EM2} = \alpha_{EM} - \sqrt{\alpha_{EM}^2 - 4\nu_{EM} + 4\psi_{EM}} \quad (77)$$

$$\Omega_{EM1} = 2\left(\psi_{EM} + \sqrt{\psi_{EM}^2 - 4\delta_{EM}}\right) \quad (78)$$

It is very interesting to notice that coefficients $a_{71}, a_{72}, a_{73}, a_{74}, a_{75}, a_{76}$ do not influence the eigenvalues of $\underline{\underline{\mathbf{A}}}_{EM}$. This means that it is irrelevant which terms in the mathematical model for the electric conduction current (equation 20) should be used in this analysis since they do not influence the eigenvalues of matrix $\underline{\underline{\mathbf{A}}}_{EM}$.

It is impossible to make any general conclusions about the values of the eigenvalues by looking at the equations (64-67). Nevertheless, eigenvalues for two simplified cases of EMHD flows can be obtained. For one-dimensional EMHD flows, we have $v_y = v_z = 0$. Under these conditions we get that $a_{22} = a_{33}$ and $a_{25} = 0$. Hence

$$\lambda_E^+ = \lambda_B^+ = \frac{\chi^E v_x + \sqrt{\chi^{E^2} v_x + 4\left(\frac{1}{\mu\varepsilon_o} - \chi^E v_x^2\right)}}{2} \quad (87)$$

$$\psi_{EM} = \sqrt[3]{Y_{EM} + \sqrt{Z_{EM}^3 + Y_{EM}^2}} + \sqrt[3]{Y_{EM} - \sqrt{Z_{EM}^3 + Y_{EM}^2}} + \frac{\nu_{EM}}{3} \tag{80}$$

$$Z_{EM} = \frac{3(\alpha_{EM}\gamma_{EM} - 4\delta_{EM}) - \nu_{EM}^2}{9} \tag{81}$$

$$Y_{EM} = \frac{\nu_{EM}(4\delta_{EM} - \alpha_{EM}\gamma_{EM})}{6} + \frac{\nu_{EM}^3}{27} - \frac{(4\nu_{EM}\delta_{EM} - \gamma_{EM}^2 - \alpha_{EM}^2\delta_{EM})}{2} \tag{82}$$

Coefficients in the fourth order characteristic polynomial are

$$\alpha_{EM} = -a_{22} - a_{33} \tag{83}$$

$$\nu_{EM} = a_{22}a_{33} - a_{26} + a_{35} \tag{84}$$

$$\gamma_{EM} = -a_{26}a_{33} - a_{22}a_{35} \tag{85}$$

$$\delta_{EM} = a_{25}a_{36} - a_{35}a_{26} \tag{86}$$

$$\lambda_E^- = \lambda_B^- = \frac{\chi^E v_x - \sqrt{\chi^{E^2} v_x + 4\left(\frac{1}{\mu\varepsilon_o} - \chi^E v_x^2\right)}}{2} \tag{88}$$

In the case of a pure electro-magnetics without any fluid motion, polarization, magnetization or electric charges ($\underline{\mathbf{v}} = \underline{\mathbf{P}} = \underline{\mathbf{M}} = q_o = 0$), these eigenvalues reduce to the eigenvalues of the classical Maxwell's equations for electro-magnetic fields in vacuum (Shang, 1993)

$$\lambda = \left\{0, \frac{1}{\sqrt{\varepsilon_o\mu_o}}, -\frac{1}{\sqrt{\varepsilon_o\mu_o}}, 0, \frac{1}{\sqrt{\varepsilon_o\mu_o}}, -\frac{1}{\sqrt{\varepsilon_o\mu_o}}\right\}^* \tag{89}$$

Notice that the non-zero eigenvalues in equation (89) are equal to the speed of light. From the equations (87-88), it is obvious that the incoming and outgoing electro-magnetic waves will be influenced by the fluid motion and the physical properties of the fluid at the inlet and exit boundaries. Nevertheless, this influence will be negligible except at extremely high fluid speeds and when the fluid is very highly ionized so that it has extremely high electric susceptibility.

After introducing the inverse similarity transformation matrix $\underline{\underline{\mathbf{S}}}_{EM}^{-1}$, the eigenmatrix $\underline{\underline{\tilde{\Lambda}}}_{EM}$ corresponding to the flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{EM}$ is

$$\underline{\underline{\tilde{\Lambda}}}_{EM} = \mathrm{diag}[0, \lambda_E^+, \lambda_E^-, 0, \lambda_B^+, \lambda_B^-, v_x] \qquad (90)$$

where $\lambda_E^+, \lambda_E^-, \lambda_B^+, \lambda_B^-$ are given by equations (72-75). For locally one-dimensional problems, wave propagation direction is well defined. For multi-dimensional problems, there is no unique direction of propagation because the coefficient matrices $\underline{\underline{\mathbf{A}}}_{EM}$, $\underline{\underline{\mathbf{B}}}_{EM}$ and $\underline{\underline{\mathbf{C}}}_{EM}$ cannot be simultaneously diagonalized. Therefore, boundary condition analysis requires that any one coordinate direction be diagonalized at a time. For most cases, this direction is the main flow direction.

Premultiplying the governing equation (54) with the inverse of the similarity matrix $\underline{\underline{\mathbf{S}}}_{EM}$ of $\underline{\underline{\mathbf{A}}}_{EM}$ gives

$$\underline{\underline{\mathbf{S}}}_{EM}^{-1} \frac{\partial \mathbf{Q}_{EM}}{\partial t} + \underline{\underline{\tilde{\Lambda}}}_{EM} \underline{\underline{\mathbf{S}}}_{EM}^{-1} \frac{\partial \mathbf{Q}_{EM}}{\partial x} + \underline{\underline{\mathbf{S}}}_{EM}^{-1} \tilde{\mathbf{H}}_{EM} = 0 \qquad (91)$$

where the i-th equation is

$$\underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \frac{\partial \mathbf{Q}_{EM}}{\partial t} + \underline{\underline{\tilde{\Lambda}}}_{i,EM} \underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \frac{\partial \mathbf{Q}_{EM}}{\partial x} + \underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \tilde{\mathbf{H}}_{EM} = 0 \qquad (92)$$

Here, left eigenvector $\underline{\underline{\mathbf{S}}}_{i,EM}^{-1}$ is the i-th row of $\underline{\underline{\mathbf{S}}}_{EM}^{-1}$ while the new source vector $\tilde{\mathbf{H}}_{EM}$ is given as

$$\tilde{\mathbf{H}}_{EM} = \underline{\underline{\mathbf{B}}}_{EM} \frac{\partial \mathbf{Q}_{EM}}{\partial y} + \underline{\underline{\mathbf{C}}}_{EM} \frac{\partial \mathbf{Q}_{EM}}{\partial z} - \mathbf{S}_{EM} \qquad (93)$$

For hyperbolic system of partial differential equations, time-dependent boundary conditions could be derived based on the principle that outgoing waves are described by characteristic equations, while the incoming waves may often be specified by a non-reflecting boundary condition (Thompson, 1987; 1990). Following Thompson's approach, the characteristic and non-reflecting boundary conditions at the inlet boundary x = a and at the outlet boundary x = b can be given by a general equation

$$\left(\underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \frac{\partial \mathbf{Q}_{EM}}{\partial t} + \mathrm{L}_{i,EM} + \underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \tilde{\mathbf{H}}_{EM} \right) \Bigg|_{x=a,b} = 0 \qquad (94)$$

where

$$\mathrm{L}_{i,EM} = \begin{cases} \underline{\underline{\tilde{\Lambda}}}_{i,EM} \underline{\underline{\mathbf{S}}}_{i,EM}^{-1} \dfrac{\partial \mathbf{Q}_{EM}}{\partial x} & \text{for outgoing waves} \\ 0 & \text{for incoming waves} \end{cases} \qquad (95)$$

The essence of his approach is that one-dimensional characteristic analysis can be performed by considering the transverse terms (in y and z directions) as a constant source term, $\tilde{\mathbf{H}}_{EM}$. It is important to realize that one-dimensional problems are quite different from multi-dimensional ones as far as the non-reflecting boundary conditions are concerned. In order to provide well-posed non-reflecting boundary conditions in multi-dimensional cases, substantial modifications must be made by taking into account the transverse terms at the boundaries (Poinsot and Lele, 1992; Hixon and Shih, 1995).

Other important points for implementing non-reflecting boundary conditions are inherently associated with the physical aspects of the problem. Specifically, there are cases where flow information propagates back from the outside of the domain into the inside through the boundaries by the incoming waves (Hagstrom and Hariharan, 1994). Thus, building a perfectly non-reflecting boundary condition might lead to an ill-posed problem. In such cases, corrections may be needed to make boundary conditions partially non-reflecting. Another physical concern is that multi-dimensionality is not fully addressed in Thompson-type non-reflecting boundary conditions. It works in the sense that the non-reflecting treatments lead to a small level of reflection, but still prevents numerical oscillations and ensures well-posedness.

Characteristic Boundary Conditions for Navier-Stokes Subsystem

Formulating non-reflecting boundary conditions for the Navier-Stokes subsystem of the EMHD system follows the same procedure as in the case of the Maxwell's subsystem. This treatment allows us to capture the important impact of electro-magnetic terms on the characteristics of the Navier-Stokes subsystem. Characteristic treatment of the Navier-Stokes subsystem of the unified EMHD system can be performed by rewriting its conservative form

$$\frac{\partial \mathbf{Q}_{NS}}{\partial t} + \frac{\partial \mathbf{E}_{NS}}{\partial x} + \frac{\partial \mathbf{F}_{NS}}{\partial y} + \frac{\partial \mathbf{G}_{NS}}{\partial z} = \mathbf{S}_{NS} \qquad (96)$$

into its non-conservative (characteristic) form

$$\frac{\partial \mathbf{Q}_{NS}}{\partial t} + \underline{\underline{\mathbf{A}}}_{NS} \frac{\partial \mathbf{Q}_{NS}}{\partial x} + \underline{\underline{\mathbf{B}}}_{NS} \frac{\partial \mathbf{Q}_{NS}}{\partial y} + \underline{\underline{\mathbf{C}}}_{NS} \frac{\partial \mathbf{Q}_{NS}}{\partial z} = \mathbf{S}_{NS} \qquad (97)$$

where the solution vector of unknowns is given as

$$\mathbf{Q}_{NS} = \left\{p/\beta, v_x, v_y, v_z, e\right\}^* \tag{98}$$

From equation (20) it can be seen that flux vector $\mathbf{E}_{NS}$ becomes

$$\mathbf{E}_{NS} = \left\{\begin{matrix} v_x \\ v_x^2 + \frac{p}{\rho} - \frac{\tau_{xx}^v + \tau_{xx}^{EM}}{\rho} - \frac{N_{EP}}{\rho} - \frac{N_{BM}}{\rho} - \frac{v_x K_x}{\rho} \\ v_x v_y - \frac{\tau_{xy}^v + \tau_{xy}^{EM}}{\rho} - \frac{v_y K_x}{\rho} \\ v_x v_z - \frac{\tau_{xz}^v + \tau_{xz}^{EM}}{\rho} - \frac{v_z K_x}{\rho} \\ e v_x + \frac{\dot{q}_x}{\rho} - \frac{I_x}{\rho} \end{matrix}\right\} \tag{99}$$

The characteristic form of Navier-Stokes subsystem influenced by the electro-magnetic effects is then obtained as

$$\underline{\underline{\mathbf{S}}}_{NS}^{-1} \frac{\partial \mathbf{Q}_{NS}}{\partial t} + \tilde{\underline{\underline{\Lambda}}}_{NS} \underline{\underline{\mathbf{S}}}_{NS}^{-1} \frac{\partial \mathbf{Q}_{NS}}{\partial x} + \underline{\underline{\mathbf{S}}}_{NS}^{-1} \tilde{\mathbf{H}}_{NS} = 0 \tag{100}$$

where the i-th equation is

$$\underline{\underline{\mathbf{S}}}_{i,NS}^{-1} \frac{\partial \mathbf{Q}_{NS}}{\partial t} + \tilde{\underline{\underline{\Lambda}}}_{i,NS} S_{i,NS}^{-1} \frac{\partial \mathbf{Q}_{NS}}{\partial x} + \underline{\underline{\mathbf{S}}}_{i,NS}^{-1} \tilde{\mathbf{H}}_{NS} = 0 \tag{101}$$

and the new source vector is

$$\tilde{\mathbf{H}}_{NS} = \underline{\underline{\mathbf{B}}}_{NS} \frac{\partial \mathbf{Q}_{NS}}{\partial y} + \underline{\underline{\mathbf{C}}}_{NS} \frac{\partial \mathbf{Q}_{NS}}{\partial z} - \mathbf{S}_{NS} \tag{102}$$

Here, the left eigenvector $\underline{\underline{\mathbf{S}}}_{i,NS}^{-1}$ is the i-th row of $\underline{\underline{\mathbf{S}}}_{NS}^{-1}$. The Jacobian matrix $\underline{\underline{\mathbf{A}}}_{NS} = \partial \mathbf{E}_{NS} / \partial \mathbf{Q}_{NS}$ then becomes

$$\underline{\underline{\mathbf{A}}}_{NS} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \beta/\rho & a_{22} & a_{23} & a_{24} & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & a_{42} & a_{43} & a_{44} & 0 \\ v_x\beta/\rho & a_{52} & a_{53} & a_{54} & v_x \end{bmatrix} \tag{103}$$

Terms related to $\underline{\underline{\mathbf{d}}}, \underline{\underline{\mathbf{d}}}^2$ and ∇T were not considered in the evaluation of coefficients of matrix $\underline{\underline{\mathbf{A}}}_{NS}$ since they are associated with first derivatives of velocity, $\underline{\mathbf{v}}$, or temperature, T. Consequently, the coefficients are

$$\begin{aligned} a_{22} = {} & 2v_x - \frac{\varepsilon_P}{\rho}\left(E_z B_y - E_y B_z\right) - \frac{\varepsilon_P}{\rho}\{B_x \\ & \left(v_z B_z + v_y B_y\right) - B_y\left(E_z + 2v_x B_y - \right. \\ & \left. v_y B_x\right) + B_z\left(E_y + v_z B_x - 2v_x B_z\right)\} \\ & - \frac{K_x}{\rho} + \frac{v_x \varepsilon_P}{\rho}\left(B_z^2 + B_y^2\right) \\ & - \frac{\alpha_{14}}{\rho\varepsilon_P}(B_y^2 + B_z^2)(P_z B_y - K_x - P_y) \\ & + \frac{\alpha_{15}}{\rho\varepsilon_P} P_x\left(B_z^2 + B_y^2\right) + \frac{\alpha_{16}}{\rho\varepsilon_P}\left(B_z^2 + B_y^2\right) \\ & \left[-\left(B_z^2 + B_y^2\right)P_x + B_x B_y P_y + B_x B_z P_z\right] \end{aligned} \tag{104}$$

$$a_{23} = -\frac{\varepsilon_P}{\rho}\left(E_xB_z - E_zB_x\right) - \frac{\varepsilon_P}{\rho}\Big\{B_y \left(v_xB_x + v_zB_z\right) + B_x\left(E_x - 2v_yB_x + v_xB_y\right) - B_z\left(E_x - v_zB_y + 2v_yB_z\right)\Big\}$$
$$- \frac{v_x}{\rho}\varepsilon_PB_xB_y - \frac{\alpha_2}{\rho\varepsilon_P}B_zP_x$$
$$+ \frac{\alpha_{14}}{\rho\varepsilon_P}\left(B_y^2B_xP_z + B_xB_yK_x + B_xB_yB_zP_y\right)$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P}\left(B_zK_x + B_xB_yP_x\right)$$
$$+ \frac{\alpha_{16}}{\rho\varepsilon_P}\Big[B_xB_y\left(B_z^2 + B_y^2\right)P_x - B_xB_yP_y - B_xB_zP_z + B_z\left(B_z^2 + B_y^2\right)K_x + \varepsilon_PB_x^2B_z\Big] \tag{105}$$

$$a_{24} = -\frac{\varepsilon_P}{\rho}\left(E_yB_x - E_xB_y\right) - \frac{\varepsilon_P}{\rho}\Big\{B_z \left(v_yB_y + v_xB_x\right) - B_x\left(E_y + 2v_zB_x - v_xB_z\right) + B_y\left(E_x + v_yB_z - 2v_zB_y\right)\Big\}$$
$$- \frac{v_x}{\rho}\varepsilon_PB_xB_z + \frac{\alpha_2}{\rho\varepsilon_P}B_yP_x$$
$$- \frac{\alpha_{14}}{\rho\varepsilon_P}\left(B_xB_zB_yP_z - B_xB_z^2P_y - B_xB_zK_x\right)$$
$$+ \frac{\alpha_{15}}{\rho\varepsilon_P}\left(B_yK_x - B_xB_zP_x\right)$$
$$- \frac{\alpha_{16}}{\rho\varepsilon_P}\Big\{B_xB_z\Big[\left(-B_z^2 - B_y^2\right)P_x + B_xB_yP_y + B_xB_zP_z\Big] + K_x\Big[\left(B_z^2 + B_y^2\right) + B_x^2B_y\Big]\Big\} \tag{106}$$

$$a_{32} = v_y + \frac{\varepsilon_Pv_y}{\rho}\left({B_y}^2 + {B_z}^2\right) - \frac{\alpha_2}{\rho\varepsilon_P}B_zP_x$$
$$+ \frac{\alpha_{14}}{\rho\varepsilon_P}\Big[\left(B_z^2 + B_y^2\right)\left(B_zP_x - B_xP_z\right) + B_xB_yK_x\Big]$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P}B_xB_yP_x - \frac{\alpha_{16}}{\rho\varepsilon_P}\Big\{\left(-B_z^2 - B_y^2\right)\Big[B_xB_yP_x - \left(B_z^2 + B_x^2\right)P_y + B_zB_yP_z\Big] + K_x\Big[\left(B_z^2 + B_x^2\right)B_z + B_zB_y^2\Big]\Big\} \tag{107}$$

$$a_{33} = v_x - \frac{1}{\rho}K_x - \frac{v_y}{\rho}\varepsilon_PB_xB_y - \frac{\alpha_2}{\rho\varepsilon_P}B_zP_y$$
$$- \frac{\alpha_{14}}{\rho\varepsilon_P}\left(B_xB_yB_zP_x + (B_x^2 + B_z^2)K_x - B_x^2B_yP_z\right)$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P}\Big[B_zK_y - \left(B_x^2 + B_z^2\right)P_x\Big]$$
$$- \frac{\alpha_{16}}{\rho\varepsilon_P}B_xB_y\Big[B_xB_yP_x - \left(B_x^2 + B_z^2\right)P_y + B_zB_yP_z\Big] \tag{108}$$

$$a_{34} = -\frac{v_y\varepsilon_P}{\rho}B_xB_z + \frac{\alpha_2}{\rho\varepsilon_P}\left(B_yP_y + B_xP_x\right)$$
$$- \frac{\alpha_{14}}{\rho\varepsilon_P}\left(B_z^2B_xP_x - B_yB_zK_x - B_x^2B_zP_z\right)$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P}\left(B_yB_zL_x - B_yK_y\right) \tag{109}$$
$$- \frac{\alpha_{16}}{\rho\varepsilon_P}\Big\{B_xB_z\Big[B_xB_yP_x - \left(B_x^2 + B_z^2\right)P_y + B_yB_zP_z\Big] + K_x\Big[B_xB_y^2 - B_x\left(B_x^2 + B_z^2\right)\Big]\Big\}$$

$$a_{42} = v_z + \frac{\varepsilon_P v_z}{\rho}\left(B_y{}^2 + B_z{}^2\right) - \frac{\alpha_2}{\rho\varepsilon_P} B_y P_x$$
$$- \frac{\alpha_{14}}{\rho\varepsilon_P}\left[\left(B_y^2 + B_z^2\right)\left(B_y P_x - B_x P_y\right) - B_x B_z K_x\right]$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P} B_x B_z P_x + \frac{\alpha_{16}}{\rho\varepsilon_P}\Big\{\left(B_y^2 + B_z^2\right)$$
$$\left[B_x B_y P_x - \left(B_x^2 + B_z^2\right)P_y + B_y B_z P_z\right]$$
$$+ K_x\left[B_y B_z^2 + \left(B_x^2 + B_y^2\right)B_y\right]\Big\} \quad (110)$$

$$a_{43} = -\frac{v_z \varepsilon_P}{\rho} B_x B_y - \frac{\alpha_2}{\rho\varepsilon_P}\left(B_z P_z - B_x P_x\right)$$
$$+ \frac{\alpha_{14}}{\rho\varepsilon_P}\left(B_x B_y^2 P_x - B_y B_x^2 P_y + B_y B_z K_x\right)$$
$$- \frac{\alpha_{15}}{\rho\varepsilon_P}\left(B_z K_z + B_z B_y P_x\right) - \frac{\alpha_{16}}{\rho\varepsilon_P}\Big\{B_x B_y \quad (111)$$
$$\left[B_x B_z P_x + B_y B_z P_y - \left(B_x^2 + B_y^2\right)P_z\right]$$
$$+ K_x\left[B_x B_z^2 + B_x\left(B_x^2 + B_y^2\right)\right]\Big\}$$

$$a_{44} = v_x - \frac{1}{\rho} K_x - \frac{v_z}{\rho}\varepsilon_P B_x B_z + \frac{\alpha_2}{\rho\varepsilon_P} B_y P_z$$
$$- \frac{\alpha_{14}}{\rho\varepsilon_P}\left[- B_x B_y B_z P_x + \left(B_x^2 + B_y^2\right)K_x + B_x^2 B_z P_y\right]$$
$$+ \frac{\alpha_{15}}{\rho\varepsilon_P}\left[B_y K_z + \left(B_x^2 + B_y^2\right)P_x\right]$$
$$- \frac{\alpha_{16}}{\rho\varepsilon_P} B_z B_x\left[B_z B_x P_x + B_y B_z P_y - \left(B_x^2 + B_y^2\right)P_z\right] \quad (112)$$

$$a_{52} = e - \frac{1}{\rho}\left(-p + \tau_{xx}^{v}\right) - \frac{\kappa_7}{\rho}\left(B_y^2 + B_z^2\right)$$
$$- \frac{1}{\rho}\left(\tau_{xx}^{EM} + v_x \frac{\partial \tau_{xx}^{EM}}{\partial v_x} + v_y \frac{\partial \tau_{xy}^{EM}}{\partial v_x} + v_z \frac{\partial \tau_{xz}^{EM}}{\partial v_x}\right) \quad (113)$$

$$a_{53} = \frac{1}{\rho}\left(\kappa_1 B_z + \kappa_7 B_x B_y\right) - \frac{1}{\rho}\tau_{xy}^{v}$$
$$- \frac{1}{\rho}\left(\tau_{xy}^{EM} + v_x \frac{\partial \tau_{xx}^{EM}}{\partial v_y} + v_y \frac{\partial \tau_{xy}^{EM}}{\partial v_y} + v_z \frac{\partial \tau_{xz}^{EM}}{\partial v_y}\right) \quad (114)$$

$$a_{54} = \frac{1}{\rho}\left(- \kappa_1 B_y + \kappa_7 B_x B_z\right) - \frac{1}{\rho}\tau_{xz}^{v}$$
$$- \frac{1}{\rho}\left(\tau_{xz}^{EM} + v_x \frac{\partial \tau_{xx}^{EM}}{\partial v_z} + v_y \frac{\partial \tau_{xy}^{EM}}{\partial v_z} + v_z \frac{\partial \tau_{xz}^{EM}}{\partial v_z}\right) \quad (115)$$

Eigenvalue vector of the Jacobian matrix $\underline{\underline{A}}_{NS}$ is

$$\lambda_{NS} = \left\{v_x, \quad \lambda_u^+, \quad \lambda_v^+, \quad \lambda_w^+, \quad \lambda_t^+\right\}^* \quad (116)$$

which can be written as a diagonal eigenvalue matrix

$$\underline{\underline{\tilde{\Lambda}}}_{NS} = \mathrm{diag}[V_x, \lambda_u^+, \lambda_v^+, \lambda_w^+, \lambda_e^+] \quad (117)$$

in which new eigenvalues $\lambda_u^+, \lambda_v^+, \lambda_w^+, \lambda_e^+$ are obtained analytically by solving a fourth order characteristic plynomial so that

$$\lambda_u^+ = -\frac{1}{4}\Phi_{NS1} + \sqrt{\frac{1}{4}\Phi_{NS1}^2 - \Omega_{NS1}} \quad (118)$$

$$\lambda_v^+ = -\frac{1}{4}\Phi_{NS1} - \sqrt{\frac{1}{4}\Phi_{NS1}^2 - \Omega_{NS1}} \quad (119)$$

$$\lambda_w^+ = -\frac{1}{4}\Phi_{NS2} + \sqrt{\frac{1}{4}\Phi_{NS2}^2 - \Omega_{NS2}} \quad (120)$$

$$\lambda_e^+ = -\frac{1}{4}\Phi_{NS2} - \sqrt{\frac{1}{4}\Phi_{NS2}^2 - \Omega_{NS2}} \tag{121}$$

$$\Phi_{NS1} = \alpha_{NS} + \sqrt{\alpha_{NS}^2 - 4\nu_{NS} + 4\psi_{NS}} \tag{122}$$

$$\Phi_{NS2} = \alpha_{NS} - \sqrt{\alpha_{NS}^2 - 4\nu_{NS} + 4\psi_{NS}} \tag{123}$$

$$\Omega_{NS1} = 2\left(\psi_{NS} + \sqrt{\psi_{NS}^2 - 4\delta_{NS}}\right) \tag{124}$$

$$\Omega_{NS2} = 2\left(\psi_{NS} - \sqrt{\psi_{NS}^2 - 4\delta_{NS}}\right) \tag{125}$$

$$\psi_{NS} = \sqrt[3]{Y_{NS} + \sqrt{Z_{NS}^3 + Y_{NS}^2}} + \sqrt[3]{Y_{NS} - \sqrt{Z_{NS}^3 + Y_{NS}^2}} + \frac{\nu_{NS}}{3} \tag{126}$$

$$Z_{NS} = \frac{3(\alpha_{NS}\gamma_{NS} - 4\delta_{NS}) - \nu_{NS}^2}{9} \tag{127}$$

$$Y_{NS} = \frac{\nu_{NS}(4\delta_{NS} - \alpha_{NS}\gamma_{NS})}{6} + \frac{\nu_{NS}^3}{27} - \frac{\left(4\nu_{NS}\delta_{NS} - \gamma_{NS}^2 - \alpha_{NS}^2\delta_{NS}\right)}{2} \tag{128}$$

in which

$$\alpha_{NS} = a_{22} + a_{33} + a_{44} \tag{129}$$

$$\nu_{NS} = a_{22}a_{33} + a_{22}a_{44} + a_{33}a_{44} - a_{34}a_{43} - a_{24}a_{42} - a_{23}a_{32} - \frac{\beta}{\rho} \tag{130}$$

$$\gamma_{NS} = -a_{34}a_{43}a_{22} - a_{22}a_{33}a_{44} - a_{24}a_{32}a_{43} + a_{24}a_{33}a_{42} + a_{23}a_{32}a_{44} - a_{23}a_{34}a_{42} + (a_{33} + a_{44})\frac{\beta}{\rho} \tag{131}$$

$$\delta_{NS} = (a_{34}a_{43} - a_{33}a_{44})\frac{\beta}{\rho} \tag{132}$$

Characteristic waves defined by the Navier-Stokes equations in the EMHD system have a great dependency on both fluid dynamics and electro-magneto-dynamics, in particular, the electro-magnetic properties of the media and electro-magnetic field quantities. In case of no electric or magnetic fields present, these eigenvalues reduce to the well-known eigenvalues of a classical Navier-Stokes system for incompressible flows, that is, $\{v_x, v_x, v_x, v_x + c, v_x - c\}$, where the equivalent local speed of sound is defined as $c = \sqrt{v_x^2 + (\beta/\rho)}$.

Following Thompson's approach, non-reflecting boundary conditions are hence formulated as

$$\left.\left(\underline{\underline{S}}_{i,NS}^{-1}\frac{\partial \mathbf{Q}_{NS}}{\partial t} + L_{i,NS} + \underline{\underline{S}}_{i,NS}^{-1}\tilde{\mathbf{H}}_{NS}\right)\right|_{x=a,b} = 0 \tag{133}$$

where

$$L_{i,NS} = \begin{cases} \tilde{\underline{\underline{\Lambda}}}_{i,NS}\underline{\underline{S}}_{i,NS}^{-1}\dfrac{\partial \mathbf{Q}_{NS}}{\partial x} & \text{for outgoing waves} \\ 0 & \text{for incoming waves} \end{cases} \tag{134}$$

In the practical implementation of Thompson-type non-reflecting boundary conditions for Navier-Stokes subsystem, the same cautions discussed for Maxwell's subsystem also apply. Furthermore, special attention should be paid in order to achieve the best performance of boundary conditions suggested by the present approach. Numerical analysis shows that the construction of non-reflecting boundary conditions should give excellent results for one-dimensional or multi-dimensional flows which do not exhibit strong transverse gradients near the boundary. However, for the more complex cases like a strong shear flow condition at the inflow or outflow boundaries, the perfect non-reflecting boundary conditions lead to ill-posed problems (Poinsot and Lele, 1992). In these cases, more sophisticated modifications will be necessary to be incorporated in the current approach.

It is of great interest to notice that EMHD flows possess entirely new families of wave modes in comparison to hydrodynamic flows. The coupling of fluid dynamics and electromagnetism yields completely original wave modes which do not occur with pure hydrodynamics or electromagnetics. Those wave modes are also quite different from that of MHD and EHD flows (Norman, et al., 1991). This will not only result in a wealth of new phenomena, but also will have implications for the numerical algorithms. One must ensure that the numerical algorithms provide for the stable and accurate propagation of all possible EMHD wave modes.

Inflow And Outflow Boundary Conditions Of Navier-Stokes Subsystem And Maxwell's Subsystem

Inflow and outflow boundary conditions for Navier-Stokes equations and Maxwell's equations can be evolved from characteristic formulations of both subsystems. The procedure to define these boundary conditions involves three steps. At a given inlet or outlet boundary:

Step 1. Construct one-dimensional characteristic wave equations from basic conservation laws on a given boundary for each subsystem.

Step 2. According to the eigenvalue vector, identify the incoming and outgoing waves through the boundary and suppress the amplitude of incoming waves by making it explicitly zero.

Step 3. Use the remaining conservation equations for outgoing waves of each subsystem combined with non-reflecting relations of incoming waves evaluated from Step 2 to compute all of the variables on the given boundary. Either time or space derivatives may be used in the numerical implementation.

The entire procedure is an extension of Thompson-type non-reflecting boundary conditions for hyperbolic equations (Thompson, 1987; 1990). In the current stage, the characteristic boundary condition formulations suggested by the non-reflecting mechanism are based on the assumption that waves for Navier-Stokes subsystem are associated with the hyperbolic and the parabolic part of the Navier-Stokes equations. In other words, those waves will be identified by the same procedure as for Euler equations and waves associated with the diffusion processes will be included. Step 2 is the key part of the entire procedure. Using the conservation equations written on the boundary as well as some reasonable information about the amplitude of incoming waves, this approach removes the ambiguity of having to choose some arbitrary "numerical" boundary conditions.

4. CONCLUSIONS

A mathematical model of Electro-Magneto-Hydrodynamic flows amenable for numerical implementation is established through the theoretical investigation of the system of governing partial differential equations and boundary condition specification methods. In particular, a system of twelve coupled partial differential equations has been formulated in its fully conservative form. This system governs three-dimensional unsteady flows of incompressible, homocompositional, electrically conducting, linearly polarizable and magnetizable fluids carrying electrically charged particles while exposed to unsteady externally applied nonuniform electric and magnetic fields. The conservative forms are suitable either for further transformations into other orthogonal and non-orthogonal coordinate systems or for direct discretization using finite difference and finite volume techniques. A systematic procedure for deriving characteristic and non-reflecting boundary conditions for the EMHD system has been developed. New features associated with fluid polarization and magnetization effects and flow kinetics are captured in the formulations of Maxwell's subsystem boundary conditions and Navier-Stokes subsystem boundary conditions. While fluid flow influences open boundary conditions of the Maxwell's subsystem negligibly, the electric and magnetic fields influence the open boundary conditions of the Navier-Stokes subsystem substantially. In practice, these features may consist of major challenges imposed on numerically implementing EMHD boundary conditions.

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