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Inverse determination of boundary conditions in multi-domain heat transfer problems

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Abstract

This paper demonstrates the capabilities of the Boundary Element Method (BEM) and the Boundary/Domain Integral Method (BDIM) in determining boundary conditions in multi-domain problems involving heat conduction and linear heat conduction/convection. The effect of the conductivity ratios between multiple regions is examined and it was found that the condition number of the solution matrix increases with an increasing conductivity ratio. The BDIM was also used to solve the thermal energy equation in a fluid given a known velocity and pressure field. An iterative boundary condition correction procedure for exit thermal boundary conditions in computational fluid dynamics analysis is discussed. A conjugate conduction/convection heat transfer problem was also formulated and demonstrated using the BDIM. The BDIM is fast and non-iterative. It does not require discretization within solid regions and the solid/fluid coupling of the heat transfer is strongly implicit.

1 Introduction

A well-posed boundary value problem is constructed using boundary data measured at a finite number of locations over every boundary of the multi-dimensional domain. If the domain contains sources, their magnitudes and locations should be known everywhere in the domain. Nevertheless, measurements and continuous monitoring of the boundary data and internal sources in the entire domain are often impractical because of the intrusive nature of the large number of sensors. Using sensors may be even impossible to achieve in practice because of the highly volatile environment on certain boundaries. The placement of thermal sensors may also be impossible because of the prohibitively small size of the domain. Thus, in many cases, we are forced to solve an ill-posed boundary value problem where no data is available on certain surfaces. Ill-posed problems involve the determination of unknown boundary conditions on inaccessible boundaries using interior measurements and over-determined measurements on accessible boundaries.

2 Steady inverse heat conduction using the BEM

Inverse heat conduction problems (IHCP) represent a subclass of ill-posed problems which have been extensively investigated. The unsteady IHCP involves estimation of the unsteady boundary heat fluxes utilizing measured interior temperature histories. Measurement data errors, as well as round-off errors, are amplified by the typical iterative unsteady IHCP algorithms as pointed out by Beck et al.[1]. This error introduced into the algorithm, either by round-off, discretization, or in the measurement data, is magnified as the solution proceeds.

To date, many of the unsteady IHCP solutions were performed for specific geometries and cannot be readily extended to complex geometries. Another basic concern is that relatively few of the IHCP techniques used in engineering provide a quantitative method for determining what effect their smoothing operations have on the actual heat conduction physics.

A simple modification to the Boundary Element Method (BEM) has been found to be a very powerful alternative to the more common unsteady IHCP methodologies by solving the steady IHCP. The BEM has been used to solve many subclasses of ill-posed heat conduction problems for multi-dimensional, multiply-connected domains, including regions with different temperature-dependent material properties. Steady heat conduction is governed by

$$\nabla \bullet (k \nabla T) = 0$$

where k is the coefficient of heat conductivity, T is the temperature, and ∇ is the spatial differentiation operator. Kirchhoff's transformation linearizes this equation and produces Laplace's equation for the heat function, u.

$$u = \int_0^T \frac{k(T)}{k_0} dT$$

The discretized boundary integral equation (BIE) of the steady heat conduction system is normally written in the following matrix form (Brebbia & Dominguez[2]).

$$[\mathbf{H}]\{\mathbf{U}\} = [\mathbf{G}]\{\mathbf{Q}\}$$

where each matrix has as many rows and columns as there are boundary nodes. For two-dimensional problems the entire boundary contour is discretized into N boundary elements connected at their endpoints between the same number of boundary nodes. In order to solve this set, all of the unknowns will be collected on the left-hand side, while all of the knowns are assembled on the right. Each coefficient matrix may be multiplied by the vector of known boundary conditions to form a vector of knowns $\{\mathbf{F}\}$. The left-hand side remains in the form $[\mathbf{A}]\{\mathbf{X}\}$. Also, additional equations may be added to the equation set if, for example, temperature or temperature gradient measurements are known at certain locations within the domain. With well-posed boundary conditions, the BEM produces a solution matrix which can be solved by a Gaussian elimination or LU decomposition matrix solver. When an ill-posed problem is encountered, the matrix generally becomes highly ill-conditioned and non-square. Most matrix solvers will not work well enough to produce a correct solution of the ill-conditioned algebraic system.

Truncated singular value decomposition (SVD) methods (Press et al.[3]) can be used to obtain a solution to a highly ill-conditioned system. The approach is somewhat similar, at least in theory, to selectively discarding eigenvalues and eigenvectors of a particular system of equations that tends to magnify errors. The singular values are the eigenvalues of the square of the matrix $[\mathbf{A}]^T[\mathbf{A}]$. For a well-conditioned matrix, these values will be roughly of the same order of magnitude. As the matrix becomes more ill-conditioned, these values become more dispersed. Eliminating very small singular values has the effect of removing those algebraic terms that, because they are dominated by noise and round-off error, produce the oscillating solution vector.

In order to determine which singular values are to be truncated, we must choose a parameter τ as a singularity threshold. Any singular value whose ratio with the largest singular value is less than this singularity threshold is zeroed out. The zeroing of a small singular value corresponds to throwing away one linear combination from the set of equations that is completely corrupted by round-off error. The choice of τ is based upon the information about the uncertainty in the BEM matrix computation, the machine's floating point precision, and the standard deviation of the measurement errors in the boundary condition data. There is a range of threshold values where the algorithm will produce a correct solution (Martin & Dulikravich[4], Dulikravich & Martin[5]).

2.1 Spherical cavity within a sphere

The inverse BEM was compared to the analytic solution for multiple concentrically-located spherical regions (Dulikravich & Martin[5]). Consider an axisymmetric solution for the temperature distribution in a spherical domain with a concentrically located spherical cavity. Two concentrically-located spherical shells were analyzed. The outermost spherical surface had a non-dimensional radius of 1.2. The interface spherical surface between the two regions had a radius of 0.9 and the inner spherical cavity surface had a radius of 0.5. The outer spherical region was given a non-dimensional coefficient of thermal conductivity of 1.0 and the non-dimensional conductivity of the inner spherical region was varied from 1 to 100. The outer surface was over-specified with a non-dimensional temperature u=1.0 and a non-dimensional heat flux from the analytic solution. Nothing was specified on the cavity surface.

Isoparametric bilinear quadrilateral panels were used to discretize the spherical boundaries. Initially, the spheres were discretized with a longitudinal-latitudinal surface grid at various levels of refinements, including 8, 10, 12 and 16 boundary elements (panels), both longitudinally and latitudinally, on the outer and inner spherical boundary. The analysis version of the BEM solved for the fluxes on the outer and inner spherical boundaries. A biased error of 2% was concentrated at the poles for all levels of grid refinement.

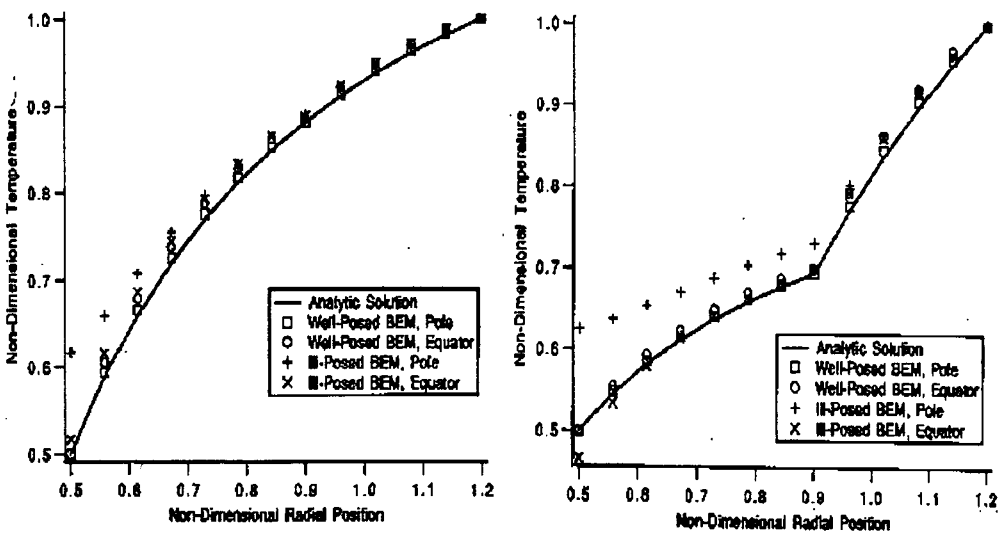

Figure 1: Radial temperature variations for the well-posed and ill-posed BEM solution. Results along radial lines and equatorial lines are compared to the analytic solution for a single region (left) and two concentric regions with conductivity ratio 1:5 (right).

The steady IHCP was then formulated by over-specifying the outer spherical boundary (applying both temperature and flux from the analytic solution), while not providing any thermal boundary conditions on the inner (cavity) spherical boundary. The BEM predicted temperatures and heat fluxes on the spherical cavity surface. The inverse results had errors of up to 16% in temperature and 3% in heat flux at the poles (see Figure 1). The error of the inverse problem was the greatest at the poles because of the type of discretization used. The quadrilateral boundary elements near the poles are nearly triangular in shape and, therefore, behave very poorly in these regions. The isoparametric shape functions become distorted and it is difficult to carry out the integration over these polar surface integrals properly due to the nature of the singular fundamental solution.

A series of numerical tests analyzed the two-domain spherical geometry for conductivity ratios between 1:1 to 1:100. The variance in the computed temperatures on the inner spherical surface were obtained for a range of SVD singularity threshold parameters in order to find τ_{opt}. It became evident that the conductivity ratio had no effect on the value of τ_{opt} which minimized the output variances. On the other hand, the error in the temperature field predicted by the inverse BEM worsened when a larger conductivity ratio was introduced. It was found that the condition number of the solution matrix becomes larger with larger disparities of thermal conductivity (Figure 1).

3 Exit boundary conditions in fluid flow

The BDIM can be used to solve the thermal energy equation in an incompressible flow-field where the velocity field is decoupled from the computed temperature field T.

$$\rho c \vec{v} \bullet \nabla T = \nabla \bullet (k \nabla T) + \Phi$$

Here, Φ is the viscous dissipation function for a known velocity field $\vec{v}$, ρ is the fluid density, c is the specific heat, and p is the pressure. The steady-state BDIE of the viscous flow thermal energy equation has been derived.

$$\frac{\theta}{2\pi} u(x) + \int_\Gamma q^*(x,\xi) u(\xi) d\Gamma = \int_\Gamma u^*(x,\xi) q(\xi) d\Gamma - \frac{1}{\alpha} \int_\Gamma v_n(\xi) u^*(x,\xi) u(\xi) d\Gamma$$
$$+ \frac{1}{\alpha} \int_\Omega \left(\vec{v}(\xi) - \vec{V} \right) \bullet \nabla u^*(x,\xi) u(\xi) d\Omega + \frac{1}{k_0} \int_\Omega u^*(x,\xi) \Phi(\xi) d\Omega$$

Note that this equation is different from the form used by Shi[6]. Here, the integration over the boundary Γ and domain Ω are with respect to the source variable ξ. The variable u is the Kirchhoff's transform of the temperature T, $q = \partial u / \partial n$, and θ is the geometric internal angle at the boundary node and is equal to 2π inside a two-dimensional domain. The function u* is the fundamental solution of Poisson's equation as the reference velocity $\vec{V} \to 0$. For high Peclet numbers, the convection/diffusion fundamental solution is recommended. The two- and three-dimensional fundamental solutions were given by Carslaw and Yaeger[7].

$$u^*(x,\xi) = \frac{1}{2\pi k} \exp\left(-\frac{\vec{V} \bullet \vec{r}}{2\alpha} \right) K_0\left(\frac{Vr}{2\alpha} \right)$$
$$u^*(x,\xi) = \frac{1}{4\pi k r} \exp\left(\frac{\vec{V} \bullet \vec{r} - Vr}{2\alpha} \right)$$

where the thermal diffusivity $\alpha = k / \rho c$ and the Peclet number is $Pe = VL / \alpha$.

The use of this system alone does not offer a realistic solution in most practical situations. For example, the temperature field will affect the velocity field in compressible flow situations, in the case where the thermal buoyancy is not negligible, and where viscosity is a function of temperature. In these instances, an iterative procedure can be implemented. It is sufficient to make an initial guess to the flow-field exit temperature boundary conditions and solve the full system of Navier-Stokes equations in order to obtain the velocity and pressure fields. Then, the steady BDIM for the energy equation uses this velocity to non-iteratively solve for the temperature field in the entire flow-field with the flow exit boundary partially or entirely unspecified. In order to compensate for the missing information, additional boundary conditions of heat flux can be over-specified at the flow-field inlet. The BDIM will compute new temperatures on the exit boundary which can be iteratively applied to the fluid flow analysis.

3.1 Poiseuille flow between parallel plates

The concept will be demonstrated on a steady, incompressible, laminar flow field with an unknown exit thermal boundary condition. Imagine two infinite flat plates with a fluid in the slot subject to a constant axial pressure gradient. In the fully-developed region, the velocity profile solution is

$$v_x = \frac{-h^2}{2\mu}\frac{dp}{dx}\left(1-\left(\frac{y}{h}\right)^2\right)$$

where μ is the dynamic viscosity coefficient.

The energy equation can be solved for the temperature field given the entire velocity field in the fluid flow. The top and bottom walls were kept at constant hot temperatures T_1=1.0. The temperature of the inlet was uniform and cold T_0=0.0. In the forward analysis mode of the BDIM, the temperature profile was considered to be fully-developed so that adiabatic boundary conditions were specified at the flow exit. The BDIM computational grid utilized 10 linear boundary elements along the boundary in the x-direction and 10 linear boundary elements along the y-direction boundary. The domain used 100 quadrilateral grid cells. The solution for the temperature profile at the mean line is plotted in Figure 3. The channel's length L was chosen so as to provide a fully-developed temperature profile at the exit.

Next, the boundary conditions at the inlet and exit of the same two parallel plates have been changed to produce an ill-posed problem. No thermal boundary condition was specified at the exit boundary, while the inlet was over-specified with both uniform temperature and a zero normal temperature gradient. The ill-conditioned BDIM system was solved using the SVD with a singularity threshold of τ=0.01. Figure 4 shows the isotherms in the channel for the case where the length of the channel was L/2 and Pe = 20.0.

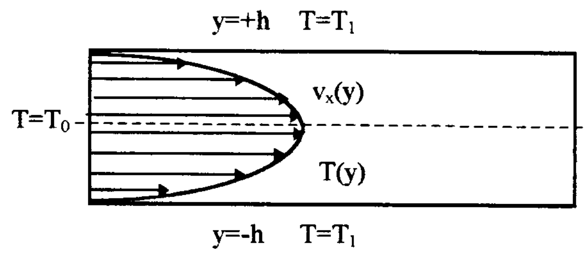

Figure 2: Poiseuille flow between parallel plates showing the velocity profile.

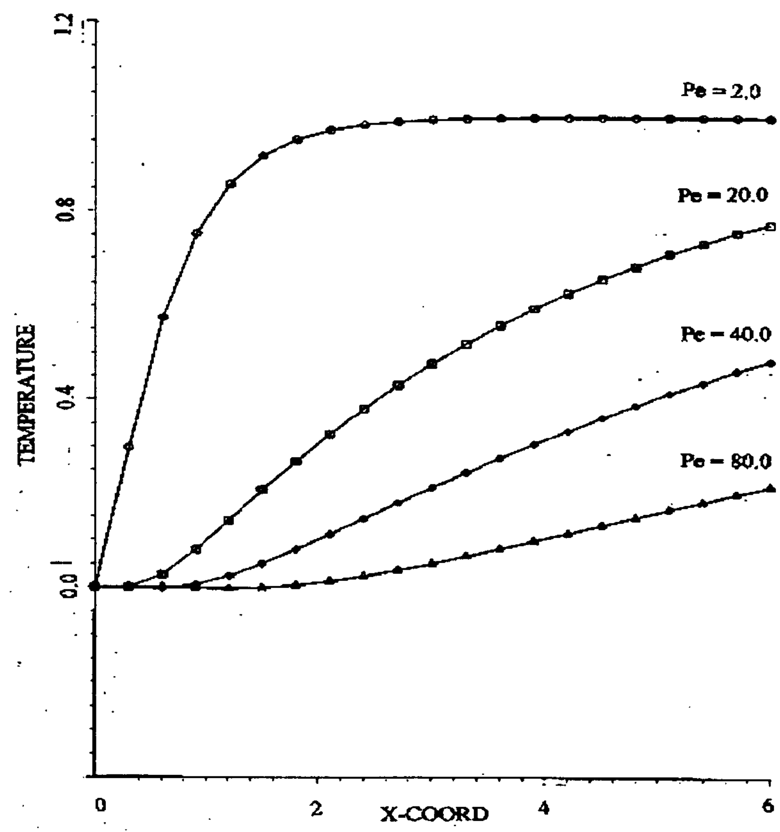

Figure 3: Temperature along the mean line for Poiseuille flow between parallel plates. Results are shown for Peclet numbers Pe = 2.0, 20.0, 40.0 and 80.0.

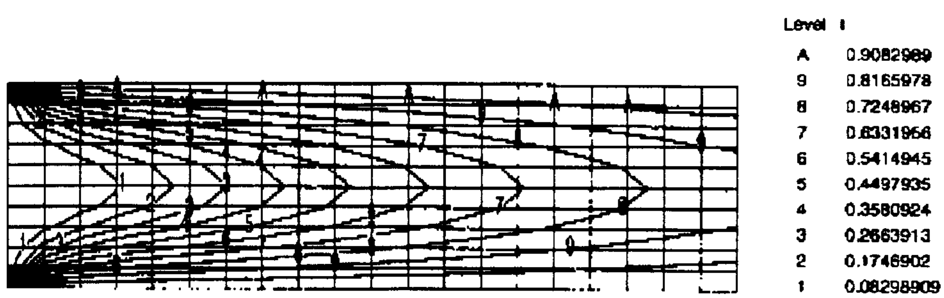

Figure 4: Temperature field for the thermal entry problem with a fully-developed Poiseuille velocity profile and an unknown exit temperature boundary condition.

4 Conjugate heat transfer

Conjugate heat transfer problems have been studied for many years. In most publications, conjugate heat transfer predictions involve an explicit coupling between the computational regions with separate analysis programs (Yeung and Liburdy[81]) such as Finite Element (FEM) and Finite Difference (FDM) algorithms for both domains. However, FEM and FDM require the generation of grids within the domain of the solid container. This can be very difficult, especially when the solid containers have complicated internal coolant flow passages.

The BEM requires minimal computational effort and eliminates the need for a computational grid within the solid since it requires only surface grids. Li and Kassab[9] used the BEM to calculate the temperature field within an internally cooled turbine blade coupled with the finite volume method for predicting the viscous fluid flow. The fluid flow and thermal convection were resolved by solving the time-dependent Navier-Stokes equations on a non-skewed, shifted periodic grid. The temperature distribution in the solid portion of the blade was determined using the steady-state BEM. The conjugate solution was obtained iteratively until convergence to the steady-state was achieved. The blade's surface temperature obtained from the solution of the Navier-Stokes system was used as the boundary condition to the BEM. The heat flux computed by the BEM was then iteratively enforced as a boundary condition to the Navier-Stokes energy equation. Hildebrand et al.[10] have presented a similar coupled FDM/BEM strategy for high-temperature, hypersonic re-entry vehicles. A complication arises in these situations because there is a lack of strict enforcement of the balance of heat fluxes predicted at the interface between the fluid and solid regions.

As an alternative, a steady-state BDIM can be used to resolve the thermal energy equations in the fluid-flow and solid regions simultaneously. A multi-domain BEM heat conduction analysis program could be implicitly coupled to a BDIM thermo-viscous energy equation solver through the compatibility relations at the solid/fluid interface boundary.

$$(T)_{Solid} = (T)_{Fluid} \qquad \left(-k\frac{\partial T}{\partial n}\right)_{Solid} = \left(k\frac{\partial T}{\partial n}\right)_{Fluid}$$

The strong coupling strategy involves bringing both computational domains together in a single solution matrix. For example, the two energy equations that govern heat convection and conduction in the fluid flow region and heat conduction in the solid region can be represented in the following matrix form.

$$\begin{bmatrix} H_F^1 & H_F^I & -G_F^I & 0 \\ 0 & H_S^I & \frac{k_S}{k_F}G_F^I & H_S^2 \end{bmatrix} \begin{Bmatrix} T_F^1 \\ T_{FS}^I \\ Q_{FS}^I \\ T_S^2 \end{Bmatrix} = \begin{bmatrix} G_F^1 & 0 \\ 0 & G_S^2 \end{bmatrix} \begin{Bmatrix} Q_F^1 \\ Q_S^2 \end{Bmatrix} + \begin{bmatrix} D_F^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} S_F \\ 0 \end{Bmatrix}$$

Here, the subscripts S and F indicate solid and fluid region quantities, respectively. The subscript FS indicates quantities at the solid/fluid interface. The $\{S_F\}$ vector entries include the source terms associated with convection and the viscous dissipation function in the thermal energy equation.

The BDIM approach has three distinct advantages: no grid discretization is required in the solid region, the strongly implicit conjugate analysis non-iteratively predicts the temperature field in the solid and fluid concurrently, and the BDIM is non-iterative. Because of these advantages, very accurate predictions of the temperature and heat flux at the solid/fluid interface surface should be obtained. In addition, ill-posed conjugate fluid flow problems can be resolved when the exit thermal boundary conditions are unknown. When the temperature field influences the velocity field, as is the case when thermal buoyancy or compressibility exists, the BDIM can serve as an iterative boundary condition correction to the flow-field analysis. Only a very small number of iterative viscous flow-field solutions is required in between the fast BDIM analyses.

4.1 Poiseuille flow between parallel plates for conjugate heat transfer

This concept will be demonstrated in the following simple example. The solid parallel plates which are separated by the fluid have been included within the computational region. Each plate has a thickness of d. The bottom edge of the bottom plate and the top edge of the top plate are kept at constant temperature (T_1=1.0). The incompressible flow-field was discretized with a 10x20 grid and each solid plate was discretized with a 5x20 grid. The conductivity ratio between the solid and the fluid is 5 to 1.

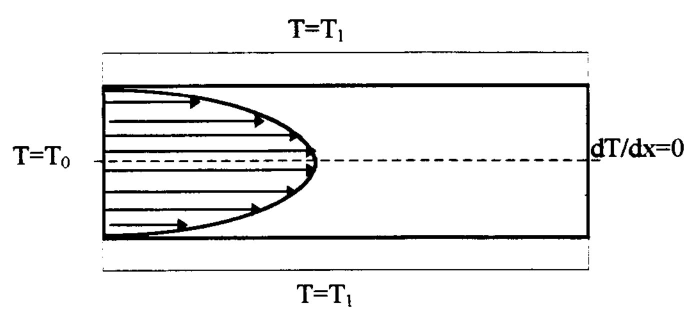

Figure 5: Poiseuille flow between parallel plates with connected solid regions.

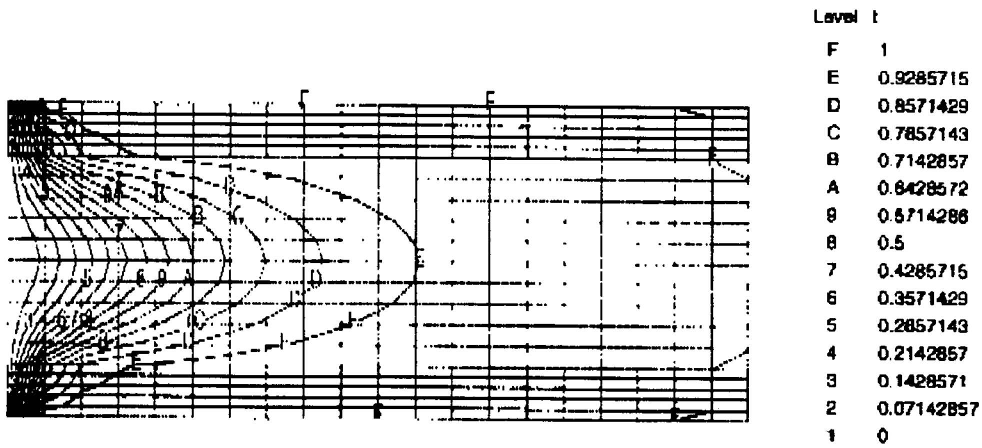

Figure 6: Temperature field for the conjugate thermal entry problem with a fully-developed Poiseuille velocity profile.

The conjugate BDIM/BEM solved for the temperature field within the fluid and two solid regions concurrently. Figure 6 shows the predicted isotherms in both regions. Notice the isotherms reaching into the solid wall plates. The grid in the solid regions is shown in the figure because it was necessary for the isotherm plotting but it was not used during the computation. The conjugate BDIM/BEM problem can obviously be solved in the inverse mode when the thermal exit boundary conditions are unavailable at the flow inlet or exit boundaries.

5 Conclusions

The inverse BEM has been developed for the determination of steady thermal boundary conditions. It has been demonstrated on a multi-domain spherical geometry made up of concentric spherical shells having different coefficients of thermal conductivity. The thermal conductivity had no effect on the SVD threshold parameter required to obtain a solution, but the temperature field predictions were worsened with the increased difference in the values of thermal conductivity. The BDIM has been shown to provide accurate results for the solution of the linear convection/diffusion equation with unspecified thermal boundaries. The procedure is non-iterative if the fluid velocity field is independent of the temperature. The BDIM appears to provide accurate results at even very high Peclet numbers. A steady-state BDIM solution of the thermal energy equation for a fluid flow has been coupled with the BEM heat conduction algorithm in a solid to solve the conjugate heat transfer problem on some simple geometries given a known fluid velocity field. The BDIM/BEM strategy is presently being extended to solve the non-linear system where the velocity field depends on the heat transfer in the fluid and solid regions simultaneously.

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