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INVERSE DETERMINATION OF TEMPERATURES AND HEAT FLUXES ON SURFACES OF 3-D OBJECTS

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ABSTRACT

A new algorithm that uses the boundary element method (BEM) has been developed for determining steady thermal boundary conditions on parts of the surfaces of three-dimensional solids where such quantities are unknown. Given temperature and heat flux on parts of the surfaces where such data is readily available, the algorithm computes, non-iteratively, the temperature field within the entire object and any unknown thermal boundary conditions on the surfaces where thermal boundary values are unavailable. The computer code is very fast and flexible in treating complex three-dimensional geometries including interior cavities and domains with different thermal properties.

INTRODUCTION

The objective of the steady-state inverse heat conduction problem is to deduce temperatures and heat fluxes on any surface or surface element where such information is unknown. In many instances it is impossible to place sensors and take measurements on a particular surface of a conducting solid due to the inaccessibility or severity of the environment on that surface. These unknown thermal boundary values may be deduced from additional temperature or heat flux measurements made within the solid or on some other surface of the solid. This problem has been given a considerable amount of attention by a variety of researchers and virtually all work has been directed to the one-dimensional transient problem [1, 2]. A characteristic of most of these inverse techniques is that they tend to produce temporal oscillations in the unknown surface thermal condition estimates that are larger than the temporal oscillations in the over-specified thermal data as it propagates through the solid [2]. In other words, the random noise due to round off errors tends to magnify as the solution proceeds and quickly produces a useless solution, especially as the distance between the surface and the over-specified information increases. A number of authors have presented various smoothing techniques for reducing this error growth, but the effect of these operations on the accuracy of the solution is not easy to evaluate.

The method presented herein does not utilize any artificial smoothing technique and is not limited to transient or one-dimensional problems. This approach is non-iterative and has been shown [3-9] to compute meaningful and accurate thermal fields in a single analysis using a straight-forward modification to the boundary element method (BEM). It should be pointed out that the inverse boundary condition method has nothing to do with the inverse domain shaping method [10].

BOUNDARY ELEMENT FORMULATION

The governing partial differential equation for steady-state heat conduction in a three-dimensional solid with a constant coefficient of thermal conductivity, and arbitrarily distributed heat sources or sinks per unit volume is

$$k\nabla^2 T(\mathbf{x}) + g(\mathbf{x}) = 0 \tag{1}$$

where k, T, **x** and g are heat conductivity coefficient, temperature, spatial position vector and internal heat source function, respectively. This elliptic partial differential equation can be subject to the Dirichlet (temperature) boundary conditions on the boundary Γ_1, $T = \overline{T}$, the Neumann (heat flux) boundary conditions on the boundary Γ_2, $-k\dfrac{\partial T}{\partial n} = \overline{Q}$, and, when a boundary is exposed to a moving fluid, the Robin (convective heat transfer) boundary conditions on the boundary Γ_3, $-k\ \partial T/\partial n = h\ (T - T_{amb})$. When an ill-posed boundary condition problem is encountered, the boundary Γ_4 has both temperature and heat flux specified and is referred to as an over-specified boundary. At the same time nothing is known on boundary Γ_5. On interface surfaces between domains made of different materials, temperatures and heat fluxes must be the same. Equation (1) can be nondimensionalized using

$$u = \frac{T - T_{min}}{\left(T_{max} - T_{min}\right)} \quad b = \frac{gl^2}{k\left(T_{max} - T_{min}\right)} \tag{2}$$

Here, l is the characteristic length scale and u is the non-dimensional temperature so that the non-dimensional heat flux is defined as $q = \frac{\partial u}{\partial n}$. the following Boundary Integral Equation (BIE) is obtained [11, 12].

$$c(\mathbf{x})u(\mathbf{x}) + \int_\Gamma q^*(\mathbf{x},y)\,u(y)\,d\Gamma = \int_\Gamma u^*(\mathbf{x},y)\,q(y)\,d\Gamma + \int_\Omega u^*(\mathbf{x},y)\,b(y)\,d\Omega \tag{3}$$

The boundary integral is singular where the observation point, **x**, is on the boundary. These singular integrals must be evaluated in the sense of the Cauchy principal value [11, 12]. Consequently, $c(\mathbf{x}) = 0.0$ when **x** is outside the domain, $c(\mathbf{x}) = 1.0$ when **x** is inside the domain and $c(\mathbf{x}) = \theta/4\pi$ when **x** is on the boundary, where θ is the internal angle at the corner between two neighboring boundary elements. The boundary was discretized into N bilinear, isoparametric, boundary elements connected at N boundary node resulting in a set of N boundary integral equations. Only this portion of the domain, Ω, that contains heat sources was discretized into N_{VC} isoparametric trilinear cells sharing both domain and boundary nodes in order to evaluate the field source integral. Since the unknowns existed only on the boundary, the set of BIE's was arranged into the following matrix form, with the geometric coefficients matrices **[H]**, [G] and [P] computed by integrating the fundamental solution distributed on the boundary and in the domain.

$$[H]\,\{U\} = [G]\,\{Q\} + [P]\,\{B\} \tag{4}$$

For a well-posed boundary value problem, every point on the boundary is given one Dirichlet, Neumann or Robin-type boundary condition, no internal temperature measurements exist, and the heat source vector **{B}** is entirely known. Once the boundary conditions are multiplied out, they can be collected on the right-hand-side and added to [P]{B} to form a vector of knowns, {F}. The left-hand side will remain in the standard form [A]{X}, having N unknowns and N equations. This system of linear algebraic equations can be solved for the unknowns on the boundary by any standard matrix solver such as Gaussian elimination or LU factorization.

If the boundary conditions in the above example are not properly applied, or if internal temperature measurements are included in the analysis, or if part or all of the heat source function is unknown, the problem becomes ill-posed. A solution may still be obtained by multiplying the known quantities in the vectors {U}, {Q} and {B} by their respective coefficient matrix columns and collecting them into the vector of knowns. The unknowns form a single vector, {X}, multiplied by a highly ill-conditioned coefficient matrix, [A], which is, in general, not square. Such matrices can be inverted using Singular Value Decomposition (SVD) methods [13, 14]

VERIFICATION OF ACCURACY OF THE COMPUTER CODE

A BEM computer program was developed using the theory discussed in the previous section. The accuracy of the three-dimensional BEM as a solution to the well-posed heat conduction problem was verified for the test case consisting of two concentric spheres with nonuniform axisymmetric temperature distributions on their surfaces that are generated by an axisymmetric heat dipole located at the center of the spheres. Surface of each sphere was discretized with 144 quadrilateral flat panels. The computed temperature derivatives normal to the surfaces of both spheres were very accurate as compared to an analytic solution and compared to the published results obtained using other numerical techniques (Tables 1 and 2). The accuracy of an inverse (ill-posed) problem was evaluated on the same configuration where the analytic values for both temperature and normal temperature derivative were specified on the surface of the outer sphere, while nothing was known on the surface of the inner sphere. The computed results were again very accurate without (Table 3) and with different levels of random noise (Table 1) incorporated in the overspecified data.

The code was then tested on a 3-D configuration consisting of a unit size cube centered at (0, 0, 0) with a centrally located cubical cavity of the size 0.4m x 0.4m x 0.4m. All six walls of the cubical cavity were at T = 100 K. Outside surface of the cube was assigned the following boundary conditions: front and back walls were adiabatic, while left and right vertical walls and the top and bottom walls were at T = 1000 K. Since a closed-form analytic solution cannot be found for this test case, we utilized a second-order accurate ADI finite difference algorithm [15] to obtain an accurate thermal field prediction for this well-posed problem. When using the finite difference analysis code and the BEM analysis code we discretized the cube with 25 x 25 x 25 uniform grid cells and the cavity with 5 x 5 x 5 uniform grid cells. Compared with the finite difference solution, the BEM analysis code had maximum local relative error of 0.2% in predicted heat fluxes on the front and back outer walls and 0.5% in predicted temperature derivative normal to the cavity walls. An ill-posed problem was then created by treating all six walls of the cubical cavity as thermally unspecified, that is, as if we do not know temperatures and heat fluxes on these boundaries. At the same time, we enforced $T = 1000$ K on all six outside walls and adiabatic conditions on the outer front and back walls thus making them the over-specified boundaries. Our inverse BEM code solved this ill-posed problem resulting in an average 5% error in the predicted cavity surface temperatures and 8% error in the predicted cavity surface heat fluxes. The errors were concentrated at the corners of the cubic cavity.

CONCLUSIONS

We have recently developed a three-dimensional version of our inverse BEM method that has the capability to determine thermal boundary conditions (temperatures and heat fluxes) on surfaces of conducting three-dimensional solid objects where such quantities are unknown. The solid can be composed of arbitrary number of arbitrarily shaped subdomains where each subdomain can have its own constant heat conductivity coefficient. Our method is very fast since it uses a non-iterative direct approach based on boundary integral method in solving steady-state inverse heat conduction problems of unknown boundary condition type. Accuracy of the computer code was tested on simple geometries where the analytic solution for steady heat conduction was known.

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Table 1: Summary of RMS errors for each scenario with concentric spheres

Scenario	**RMS Error**
Forward Solution	0.664%
Inverse Solution, Zero Noise	0.987%
Inverse Solution, 5% Noise	3.741%
Inverse Solution, 25% Noise	5.466%
Inverse Solution, 100% Noise	11.800%

Table 2: Results for analysis (well-posed) problem for a case with concentric spheres

Researchers and Discretization Level	PSU BEM, 156 nodes	Throne and Olson (1994) FEM, 3374 nodes	Pilkington *et al.* (1987) FEM, 754 nodes	Pilkington *et al.* (1987) BEM[1]
Percent RMS Error	0.664%	0.002%	0.8%	1.4%

[1] Unspecified discretization level

Table 3: Results for inverse (ill-posed) problem with zero input noise for a case with concentric spheres

Researchers and Discretization Level	PSU BEM, 156 nodes	Throne and Olson (1994) FEM, 3374 nodes	Pilkington *et al.* (1987) FEM, 754 nodes	Pilkington *et al.* (1987) BEM[2]
Percent RMS Error	0.987%	0.004%	1.2%	1.6%