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FULLY CONSERVATIVE FORMS OF A SYSTEM OF UNIFIED ELECTRO-MAGNETO-HYDRODYNAMIC EQUATIONS

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ABSTRACT

A system of partial differential equations governing unsteady multidimensional electro-magneto-hydrodynamic flows has been rewritten in its fully conservative (divergence-free) form. Linear polarization and magnetization effects were allowed while assuming an incompressible fluid and a single type of electrically charged particles in the flow. Both vector operator formulation and a scalar Cartesian coordinate system formulation have been derived in detail. The resulting fully conservative forms are suitable for further transformations to other coordinate systems or for direct discretization leading to their numerical integration.

NOMENCLATURE

$\underline{B}$	= magnetic flux density vector, kg A^{-1} s^{-2}
$\underline{\underline{d}} = \frac{1}{2}(\nabla\underline{v} + \nabla\underline{v}^T)$	= rate of deformation tensor, s^{-1}
$\underline{D} = \varepsilon_o \underline{E} + \underline{P}$	= electric displacement field vector, A s m^{-2}
e	= internal energy per unit mass, m^2 s^{-2}
$e = e + \frac{1}{2}\underline{v}\cdot\underline{v} + \underline{g}\cdot\underline{r}$	= total energy per unit mass, m^2 s^{-2}
$\underline{E}$	= electric field vector, kg m s^{-3} A^{-1}, or V m^{-1}
$\underline{\mathcal{E}} = \underline{E} + \underline{v}\times\underline{B}$	= electromotive intensity vector, kg m s^{-3} A^{-1}
$\underline{f}$	= mechanical body force vector per unit mass, m s^{-2}
$\underline{f}^{EM}$	= electromagnetic body force vector per unit volume, kg m^{-2} s^{-2}
$\underline{g}$	= acceleration due to gravity, m s^{-2}
h	= heat source or sink per unit mass, m^2 s^{-3}
$\underline{H} = \underline{B}/\mu_o - \underline{M}$	= magnetic field intensity vector, A m^{-1}
$\underline{J} = \underline{J}_c + \underline{J}_d$	= electric current density vector, A m^{-2}
$\underline{J}_c$	= electric conduction current vector, A m^{-2}
$\underline{J}_d = \underline{v}\, q_o$	= electric drift current vector, A m^{-2}
$\underline{M}$	= total magnetization vector per unit volume, A m^{-1}
$\underline{\mathcal{M}} = \underline{M} + \underline{v}\times\underline{P}$	= magnetomotive intensity vector per unit volume, A m^{-1}
p	= pressure, kg m^{-1} s^{-2}
$\underline{P}$	= total polarization vector per unit volume, A s m^{-2}
q_o	= total or free electric charge per unit volume, A s m^{-3}
$\underline{q}$	= heat flux vector, kg s^{-3}
$\underline{r}$	= position vector, m
$\underline{\underline{t}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}^v + \underline{\underline{\tau}}^{EM}$	= fluid stress tensor, kg m^{-1} s^{-2}
$\underline{v}$	= fluid velocity vector, m s^{-1}

GREEK SYMBOLS

β	= Chorin's (1967) artifical compressibility coefficient
ε	= dielectric constant or electric permittivity, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_o = 8.854\times10^{-12}$	= vacuum dielectric constant or electric permittivity, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_r = \varepsilon/\varepsilon_o$	= relative electric permittivity
ρ	= fluid density, kg m^{-3}
$\underline{\underline{\tau}}^v$	= viscous stress tensor, kg m^{-1} s^{-2}
$\underline{\underline{\tau}}^{EM}$	= electromagnetic stress tensor, kg m^{-1} s^{-2}
μ	= magnetic permeability coefficient, kg m A^{-2} s^{-2}
$\mu_o = 4\pi\times10^{-7}$	= magnetic permeability of vacuum, kg m A^{-2} s^{-2}

$\mu_r = \mu / \mu_o$ = relative magnetic permeability
$\chi^E = \varepsilon_r - 1$ = electric susceptibility
$\chi^M = \mu_r - 1$ = magnetic susceptibility

1. INTRODUCTION

Magneto-hydrodynamics (MHD) is a phenomena associated with the interaction of an externally applied magnetic field with an incompressible electrically conducting moving fluid. No electric fields or electrically charged particles are allowed in this model. Electro-hydrodynamics (EHD) is a phenomena associated with the interaction of an externally applied electric field with an incompressible moving fluid that contains electrically charged particles. No magnetic fields are allowed in this model. (Stuetzer 1962; Erigen and Maugin 1990a; 1990b).

When both magnetic and electric fields are simultaneously applied to an incompressible electrically conducting moving fluid containing electrically charged particles, the phenomena is called Electro-Magneto-Hydrodynamics (EMHD). Equations governing MHD, EHD, and EMHD have been summarized in our previous works (Dulikravich and Lynn 1995a; 1995b; 1996a; 1996b). These equations were not written in their fully conservative forms which are required when performing numerical integration especially if strong gradients of dependent variables are expected to exist in the flow-field.

The objective of this paper is to rewrite the system of partial deferential equations governing EMHD in a fully conservative (divergence-free) vector operator form and in a Cartesian coordinate system scalar form. These forms can the be used directly in the finite difference or finite volumn discretization of the EMHD system and its iterative integration process.

In order to include some of the important effects of polarization and magnetization of the moving fluid, we will assume that fluid is incompressible, homocompositional and that it has linear polarization and linear magnetization properties. In addition, frequency of the applied electric and magnetic fields should be limited to less than 1000 HZ for this mathematical model to be realistic. These are the only assumptions to be used in this work.

2. GOVERNING SYSTEM OF EQUATIONS

The full system of equations governing unified EMHD flow consists of the Maxwell's equations governing electro-magnetism, the Navier-Stokes equations governing fluid flow, and constitutive equations describing material behavior. Maxwell's equations are the system of linear differential equations governing electro-magnetic fields. They are given as

Ampere-Maxwell's law

$$\frac{\partial \underline{\mathbf{D}}}{\partial t} - \nabla \times \left(\frac{\underline{\mathbf{B}}}{\mu_0} - \underline{\mathbf{M}} \right) = -\underline{\mathbf{J}} \tag{1}$$

Faraday's law

$$\frac{\partial \underline{\mathbf{B}}}{\partial t} + \nabla \times \underline{\mathbf{E}} = 0 \tag{2}$$

Conservation of electric charges

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \tag{3}$$

Thus, the electro-magnetic phenomena will be governed by a system comprising of Eqs. (1), (2) and (3). This represents a system of seven scalar partial differential equations in the case of a three dimensional space. Notice that the conservation of electric charges represents combination of Ampere-Maxwell's law and Gauss' law given as

$$\nabla \cdot \underline{\mathbf{D}} = q_o \tag{4}$$

Detailed descriptions of these equations can be found in any number of texts (Dulikravich and Lynn, 1996a; 1996b). The equations of motion governing EMHD flow are the Navier-Stokes relations into which electromagnetic effects have been included. A rigorous derivation of these equations for electro-magnetic fluids is completed by Eringen and Maugin (1990). The expanded Navier-Stokes equations are

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\mathbf{v}}) = 0 \tag{5}$$

Conservation of linear momentum

$$\frac{\partial(\rho \underline{\mathbf{v}})}{\partial t} + \nabla \cdot \left(\underline{\mathbf{v}} \rho \underline{\mathbf{v}} - \underline{\underline{\mathbf{t}}} \right) - \rho \underline{\mathbf{f}} - \underline{\mathbf{f}}^{EM} = 0 \tag{6}$$

where the electromagnetic force per unit volume is

$$\underline{\mathbf{f}}^{EM} = q_o \underline{\mathbf{E}} + \underline{\mathbf{J}} \times \underline{\mathbf{B}} + (\nabla \underline{\mathbf{E}}) \cdot \underline{\mathbf{P}} + (\nabla \underline{\mathbf{B}}) \cdot \underline{\mathbf{M}} + \nabla \cdot (\underline{\mathbf{v}} \underline{\mathbf{P}} \times \underline{\mathbf{B}}) + \frac{\partial}{\partial t} (\underline{\mathbf{P}} \times \underline{\mathbf{B}}) \tag{7}$$

Conservation of energy

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \underline{\mathbf{v}}) - \nabla \cdot \left(\underline{\underline{\mathbf{t}}} \cdot \underline{\mathbf{v}} \right) + \nabla \cdot \underline{\mathbf{q}} - \rho h - \rho \underline{\mathcal{E}} \cdot \frac{D\left(\frac{\underline{\mathbf{P}}}{\rho} \right)}{Dt} + \underline{\mathcal{M}} \cdot \frac{D\underline{\mathbf{B}}}{Dt} - \underline{\mathbf{J}}_c \cdot \underline{\mathcal{E}} = 0 \tag{8}$$

The objective is to remove all unsteady terms from the right-hand sides of the governing equations as the first step in forming a fully conservative system of twelve coupled partial differential equations governing three-dimensional EMHD problems.

3. EVALUATING THE TERM $\frac{\partial \underline{\mathbf{P}}}{\partial t}$

We will first derive the unsteady term $\frac{\partial \underline{\mathbf{P}}}{\partial t}$ that occurs in the energy conservation so that it can be expressed as a function of steady-state terms only. We will allow only for linear polarization and linear magnetization given as

$$\underline{\mathbf{P}} = \varepsilon_0 \chi^E \underline{\mathcal{E}} \tag{9}$$

$$\underline{\mathcal{M}} = \frac{\chi^M}{\mu_0(1+\chi^M)}\underline{\mathbf{B}} \tag{10}$$

In addition, we will treat the fluid as homocompositional and incompressible.

The Ampere-Maxwell's Law can be rewritten as

$$\frac{\partial \underline{\mathbf{P}}}{\partial t}\cdot\underline{\mathbf{B}} = -\varepsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\cdot\underline{\mathbf{B}} + \left[\nabla\times\left(\frac{\underline{\mathbf{B}}}{\mu_0} - \underline{\mathbf{M}}\right)\right]\cdot\underline{\mathbf{B}} - \underline{\mathbf{J}}\cdot\underline{\mathbf{B}} \tag{11}$$

Linear constitutive relation (Eq. 9) implies that

$$\frac{\partial \underline{\mathbf{P}}}{\partial t} = \varepsilon_0\chi^E\frac{\partial \underline{\mathbf{E}}}{\partial t} + \varepsilon_0\chi^E\frac{\partial(\underline{\mathbf{v}}\times\underline{\mathbf{B}})}{\partial t} \tag{12}$$

Using the chain rule, we get that

$$\frac{\partial(\underline{\mathbf{v}}\times\underline{\mathbf{B}})}{\partial t} = \frac{\partial \underline{\mathbf{v}}}{\partial t}\times\underline{\mathbf{B}} + \underline{\mathbf{v}}\times\frac{\partial \underline{\mathbf{B}}}{\partial t} \tag{13}$$

Utilizing the conservation of linear momentum (Eq. 6) and Faraday's law (Eq. 2), it follows that for incompressible flows

$$\frac{\partial(\underline{\mathbf{v}}\times\underline{\mathbf{B}})}{\partial t} = \underline{\mathbf{G}}_v\times\underline{\mathbf{B}} + \frac{1}{\rho}\underline{\mathbf{R}}\times\underline{\mathbf{B}} - \frac{1}{\rho}[\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})]\times\underline{\mathbf{B}} + \frac{1}{\rho}\left[\left(\frac{\partial \underline{\mathbf{P}}}{\partial t}\cdot\underline{\mathbf{B}}\right)\underline{\mathbf{B}} - (\underline{\mathbf{B}}\cdot\underline{\mathbf{B}})\frac{\partial \underline{\mathbf{P}}}{\partial t}\right] + (\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}} \tag{14}$$

where the vector identity

$$\left(\frac{\partial \underline{\mathbf{P}}}{\partial t}\times\underline{\mathbf{B}}\right)\times\underline{\mathbf{B}} = \left[\left(\frac{\partial \underline{\mathbf{P}}}{\partial t}\cdot\underline{\mathbf{B}}\right)\underline{\mathbf{B}} - (\underline{\mathbf{B}}\cdot\underline{\mathbf{B}})\frac{\partial \underline{\mathbf{P}}}{\partial t}\right] \tag{15}$$

was used. Here

$$\underline{\mathbf{G}}_v = -\frac{1}{\rho}\nabla\cdot\left(\underline{\mathbf{v}}\rho\underline{\mathbf{v}} - \underline{\underline{\mathbf{t}}}\right) \tag{16}$$

$$\underline{\mathbf{R}} = \rho\underline{\mathbf{f}} + q_0\underline{\mathbf{E}} + \underline{\mathbf{J}}\times\underline{\mathbf{B}} + (\nabla\underline{\mathbf{E}})\cdot\underline{\mathbf{P}} + (\nabla\underline{\mathbf{B}})\cdot\underline{\mathbf{M}} + \nabla\cdot(\underline{\mathbf{v}}(\underline{\mathbf{P}}\times\underline{\mathbf{B}})) \tag{17}$$

By adding Eqs. (11) and (12) and notice that

$$\left(\frac{\partial \underline{\mathbf{v}}}{\partial t}\times\underline{\mathbf{B}}\right)\cdot\underline{\mathbf{B}} = 0$$

it follows that

$$\frac{\partial \underline{\mathbf{P}}}{\partial t}\cdot\underline{\mathbf{B}} = \frac{\chi^E}{1+\chi^E}\underline{\mathbf{S}}_{\mathbf{B}}\cdot\underline{\mathbf{B}} + \frac{\varepsilon_0\chi^E}{1+\chi^E}[(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}]\cdot\underline{\mathbf{B}} \tag{18}$$

where

$$\underline{\mathbf{S}}_{\mathbf{B}} = \nabla\times\left(\frac{\underline{\mathbf{B}}}{\mu_0} - \underline{\mathbf{M}}\right) - \underline{\mathbf{J}} \tag{19}$$

Upon substituting Eq. (18) into Eq. (14), the relationship between $\frac{\partial(\underline{\mathbf{v}}\times\underline{\mathbf{B}})}{\partial t}$ and $\frac{\partial \underline{\mathbf{P}}}{\partial t}$ is obtained

$$\begin{aligned}\frac{\partial(\underline{\mathbf{v}}\times\underline{\mathbf{B}})}{\partial t} = {} & \underline{\mathbf{G}}_v\times\underline{\mathbf{B}} + \frac{1}{\rho}\underline{\mathbf{R}}\times\underline{\mathbf{B}} - \frac{1}{\rho}[\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})]\times\underline{\mathbf{B}} \\ & + \frac{\chi^E}{\rho(1+\chi^E)}(\underline{\mathbf{S}}_{\mathbf{B}}\cdot\underline{\mathbf{B}})\underline{\mathbf{B}} + \frac{\varepsilon_0\chi^E}{\rho(1+\chi^E)}[((\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}})\cdot\underline{\mathbf{B}}]\underline{\mathbf{B}} \\ & - \frac{1}{\rho}(\underline{\mathbf{B}}\cdot\underline{\mathbf{B}})\frac{\partial \underline{\mathbf{P}}}{\partial t} + (\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\end{aligned} \tag{20}$$

Thus, adding Eqs. (11) and (12) with the help of Eq. (20), we finally get

$$\begin{aligned}\frac{\partial \underline{\mathbf{P}}}{\partial t} = {} & \Gamma_\sigma\underline{\mathbf{G}}_v\times\underline{\mathbf{B}} + \frac{\Gamma_\sigma}{\rho}\underline{\mathbf{R}}\times\underline{\mathbf{B}} - \frac{\Gamma_\sigma}{\rho}[\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})]\times\underline{\mathbf{B}} \\ & + \frac{\chi^E\Gamma_\sigma}{\rho(1+\chi^E)}(\underline{\mathbf{S}}_{\mathbf{B}}\cdot\underline{\mathbf{B}})\underline{\mathbf{B}} + \frac{\varepsilon_0\chi^E\Gamma_\sigma}{\rho(1+\chi^E)}[((\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}})\cdot\underline{\mathbf{B}}]\underline{\mathbf{B}} \\ & + \Gamma_\sigma(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}} + \frac{\Gamma_\sigma}{\varepsilon_0}\underline{\mathbf{S}}_{\mathbf{B}}\end{aligned} \tag{21}$$

where

$$\Gamma_\sigma = \frac{\varepsilon_0\chi^E}{1+\chi^E+\frac{\varepsilon_0\chi^E\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}}{\rho}} \tag{22}$$

4. EVALUATING THE TERM $\frac{\partial(\underline{\mathbf{P}}\times\underline{\mathbf{B}})}{\partial t}$

This term occurs in the linear momentum equation. From the chain rule, it follows that

$$\frac{\partial(\underline{\mathbf{P}}\times\underline{\mathbf{B}})}{\partial t} = \frac{\partial \underline{\mathbf{P}}}{\partial t}\times\underline{\mathbf{B}} + \underline{\mathbf{P}}\times\frac{\partial \underline{\mathbf{B}}}{\partial t} \tag{23}$$

Substituting for $\frac{\partial \underline{\mathbf{P}}}{\partial t}$ from Eq. (21) and for $\frac{\partial \underline{\mathbf{B}}}{\partial t}$ from Eq. (2) yields

$$\frac{\partial(\underline{\mathbf{P}}\times\underline{\mathbf{B}})}{\partial t}=\Gamma_\sigma\left(\underline{\mathbf{G}_v}\times\underline{\mathbf{B}}\right)\times\underline{\mathbf{B}}+\frac{\Gamma_\sigma}{\rho}\left(\underline{\mathbf{R}}\times\underline{\mathbf{B}}\right)\times\underline{\mathbf{B}}-\frac{\Gamma_\sigma}{\rho}$$
$$\left[\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{B}}\right]\times\underline{\mathbf{B}}+\Gamma_\sigma\left[(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\right]\times\underline{\mathbf{B}}$$
$$+\frac{\Gamma_\sigma}{\varepsilon_0}\underline{\mathbf{S}_B}\times\underline{\mathbf{B}}+(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{P}} \qquad \mathbf{(24)}$$

since $\underline{\mathbf{B}}\times\underline{\mathbf{B}}=0$

5. FULLY CONSERVATIVE FORM OF GOVERNING EQUATIONS FOR EMHD SYSTEM

5.1 Vector form of fully conservative governing equations for EMHD

Upon substituting Eqs. (21) and (24) in Maxwell's sub-system and Navier-Stokes' sub-system, a compact vector form of EMHD is obtained

$$\frac{\partial\underline{\mathbf{E}}}{\partial t}-\nabla\times\left[\frac{1}{\varepsilon_0\mu_0\left(1+\chi^M\right)}\underline{\mathbf{B}}+\chi^E\underline{\mathbf{v}}\times(\underline{\mathbf{E}}+\underline{\mathbf{v}}\times\underline{\mathbf{B}})\right]=\underline{\mathbf{S}^E} \qquad \mathbf{(25)}$$

$$\frac{\partial\underline{\mathbf{B}}}{\partial t}+\nabla\times\underline{\mathbf{E}}=\underline{\mathbf{0}} \qquad \mathbf{(26)}$$

$$\frac{\partial q_0}{\partial t}+\nabla\cdot\underline{\mathbf{J}}=0 \qquad \mathbf{(27)}$$

$$\nabla\cdot\underline{\mathbf{v}}=0 \qquad \mathbf{(28)}$$

$$\frac{\partial\underline{\mathbf{v}}}{\partial t}+\nabla\cdot\left(\underline{\mathbf{v}}\underline{\mathbf{v}}-\frac{1}{\rho}\underline{\underline{\mathbf{t}}}\right)-\frac{1}{\rho}(\nabla\underline{\mathbf{E}})\cdot\underline{\mathbf{P}}$$
$$-\frac{1}{\rho}(\nabla\underline{\mathbf{B}})\cdot\underline{\mathbf{M}}-\frac{1}{\rho}\nabla\cdot(\underline{\mathbf{v}}\underline{\mathbf{P}}\times\underline{\mathbf{B}})=\underline{\mathbf{S}^v} \qquad \mathbf{(29)}$$

$$\frac{\partial e}{\partial t}+\nabla\cdot(e\underline{\mathbf{v}})-\frac{1}{\rho}\left\{\nabla\cdot\left(\underline{\underline{\mathbf{t}}}\cdot\underline{\mathbf{v}}\right)-\nabla\cdot\underline{\mathbf{q}}\right\}=S^n \qquad \mathbf{(30)}$$

where the source terms are

$$\underline{\mathbf{S}^E}=\frac{1}{\varepsilon_0}\left\{-\underline{\mathbf{J}}-\Gamma_\sigma\underline{\mathbf{G}_v}\times\underline{\mathbf{B}}-\frac{\Gamma_\sigma}{\rho}\underline{\mathbf{R}}\times\underline{\mathbf{B}}+\frac{\Gamma_\sigma}{\rho}\left(\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\right)\times\underline{\mathbf{B}}\right.$$
$$-\frac{\chi^E\Gamma_\sigma}{\rho\left(1+\chi^E\right)}\left(\underline{\mathbf{S}_B}\cdot\underline{\mathbf{B}}\right)\underline{\mathbf{B}}-\frac{\varepsilon_0\chi^E\Gamma_\sigma}{\rho\left(1+\chi^E\right)}\left[\left((\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\right)\cdot\underline{\mathbf{B}}\right]\underline{\mathbf{B}}$$
$$\left.-\Gamma_\sigma(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}-\frac{\Gamma_\sigma}{\varepsilon_0}\underline{\mathbf{S}_B}\right\} \qquad \mathbf{(31)}$$

$$\underline{\mathbf{S}^v}=\underline{\mathbf{f}}+\frac{1}{\rho}\left\{q_0\underline{\mathbf{E}}+\underline{\mathbf{J}}\times\underline{\mathbf{B}}-\frac{\Gamma_\sigma}{\rho}\left[\left(\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\right)\times\underline{\mathbf{B}}\right]\times\underline{\mathbf{B}}\right.$$
$$+\Gamma_\sigma\left(\underline{\mathbf{G}_v}\times\underline{\mathbf{B}}\right)\times\underline{\mathbf{B}}+\frac{\Gamma_\sigma}{\rho}\left(\underline{\mathbf{R}}\times\underline{\mathbf{B}}\right)\times\underline{\mathbf{B}}$$
$$\left.+\Gamma_\sigma\left[(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\right]\times\underline{\mathbf{B}}+(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{P}}+\frac{\Gamma_\sigma}{\varepsilon_0}\underline{\mathbf{S}_B}\times\underline{\mathbf{B}}\right\} \qquad \mathbf{(32)}$$

$$S^n=h+\frac{1}{\rho}\left\{(\underline{\mathbf{E}}+\underline{\mathbf{v}}\times\underline{\mathbf{B}})\cdot\left[(\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{P}}+\Gamma_\sigma\underline{\mathbf{G}_v}\times\underline{\mathbf{B}}+\frac{\Gamma_\sigma}{\rho}\underline{\mathbf{R}}\times\underline{\mathbf{B}}\right.\right.$$
$$-\frac{\Gamma_\sigma}{\rho}\left(\underline{\mathbf{P}}\times(\nabla\times\underline{\mathbf{E}})\right)\times\underline{\mathbf{B}}+\frac{\chi^E\Gamma_\sigma}{\rho\left(1+\chi^E\right)}\left(\underline{\mathbf{S}_B}\cdot\underline{\mathbf{B}}\right)\underline{\mathbf{B}}$$
$$+\frac{\varepsilon_0\chi^E\Gamma_\sigma}{\rho\left(1+\chi^E\right)}\left(\left((\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}\right)\cdot\underline{\mathbf{B}}\right)\underline{\mathbf{B}}+\Gamma_\sigma(\nabla\times\underline{\mathbf{E}})\times\underline{\mathbf{v}}$$
$$\left.\left.+\frac{\Gamma_\sigma}{\varepsilon_0}\underline{\mathbf{S}_B}\right]-\frac{\chi^M}{\mu_0\left(1+\chi^M\right)}\underline{\mathbf{B}}\cdot\left[(\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{B}}-\nabla\times\underline{\mathbf{E}}\right]+\underline{\mathbf{J}_c}\cdot(\underline{\mathbf{E}}+\underline{\mathbf{v}}\times\underline{\mathbf{B}})\right\} \qquad \mathbf{(33)}$$

5.2 Fully conservative form of EMHD equations in Cartesian coordinates

The EMHD system of equations (Eqs. 25-30) can be now written in a general conservative form in terms of (x,y,z) orthogonal coordinate system as

$$\frac{\partial\underline{\mathbf{Q}}}{\partial t}+\frac{\partial\underline{\mathbf{E}}}{\partial x}+\frac{\partial\underline{\mathbf{F}}}{\partial y}+\frac{\partial\underline{\mathbf{G}}}{\partial z}=\underline{\mathbf{S}} \qquad \mathbf{(34)}$$

The solution vector $\underline{\mathbf{Q}}$ is given as

$$\underline{\mathbf{Q}}=\begin{pmatrix}E_x\\E_y\\E_z\\B_x\\B_y\\B_z\\q_0\\p/\beta\\V_x\\V_y\\V_z\\e\end{pmatrix} \qquad \mathbf{(35)}$$

where Chorin's (1967) artificial compressibility was used to create the unsteady term in the mass conservation. The flux vectors are defined as

$$
\underline{\mathbf{E}}=\begin{pmatrix}
0\\
H_z\\
-H_y\\
0\\
-E_z\\
E_y\\
J_x\\
V_x\\
V_x^2+\frac{1}{\rho}\left(p-\tau_{xx}-\tau_{xx}^{EM}-\varepsilon_0\chi^E L_{EP}-J_{BM}-V_xT_x\right)\\
V_xV_y-\frac{1}{\rho}\left(\tau_{xy}+\tau_{xy}^{EM}+V_yT_x\right)\\
V_xV_z-\frac{1}{\rho}\left(\tau_{xz}+\tau_{xz}^{EM}+V_zT_x\right)\\
eV_x+\frac{1}{\rho}\left(q_x-I_x\right)
\end{pmatrix} \quad \textbf{(36)}
$$

$$
\underline{\mathbf{F}}=\begin{pmatrix}
-H_z\\
0\\
H_x\\
E_z\\
0\\
-E_x\\
J_y\\
V_y\\
V_xV_y-\frac{1}{\rho}\left(\tau_{xy}+\tau_{xy}^{EM}+V_xT_y\right)\\
V_y^2+\frac{1}{\rho}\left(p-\tau_{yy}-\tau_{yy}^{EM}-\varepsilon_0\chi^E L_{EP}-J_{BM}-V_yT_y\right)\\
V_yV_z-\frac{1}{\rho}\left(\tau_{yz}+\tau_{yz}^{EM}+V_ZT_y\right)\\
eV_y+\frac{1}{\rho}\left(q_y-I_y\right)
\end{pmatrix} \quad \textbf{(37)}
$$

$$
\underline{\mathbf{G}}=\begin{pmatrix}
H_y\\
-H_x\\
0\\
-E_y\\
E_x\\
0\\
J_z\\
V_z\\
V_xV_z-\frac{1}{\rho}\left(\tau_{xz}+\tau_{xz}^{EM}+V_xT_z\right)\\
V_yV_z-\frac{1}{\rho}\left(\tau_{yz}+\tau_{yz}^{EM}+V_yT_z\right)\\
V_z^2+\frac{1}{\rho}\left(p-\tau_{zz}-\tau_{zz}^{EM}-\varepsilon_0\chi^E L_{EP}-J_{BM}-V_zT_z\right)\\
eV_z+\frac{1}{\rho}\left(q_z-I_z\right)
\end{pmatrix} \quad \textbf{(38)}
$$

and the vector of source term is given by

$$
\underline{\mathbf{S}}=\begin{pmatrix}
\underline{\mathbf{S}}_x^E\\
\underline{\mathbf{S}}_y^E\\
\underline{\mathbf{S}}_z^E\\
0\\
0\\
0\\
0\\
0\\
\underline{\mathbf{S}}_x^V\\
\underline{\mathbf{S}}_y^V\\
\underline{\mathbf{S}}_z^V\\
S^n
\end{pmatrix} \quad \textbf{(39)}
$$

where

$$
L_{EP}=E_x(E_x+V_yB_z-V_zB_y)+E_y(E_y+V_zB_x-V_xB_z)
+E_z(E_z+V_xB_y-V_yB_x) \quad \textbf{(40)}
$$

$$
\begin{aligned}
J_{BM}=&B_x\left\{\frac{\chi_M}{\mu_0(1+\chi^M)}B_x+\varepsilon_0\chi^E\left[V_z(E_y+V_zB_x-V_xB_z)-V_y(E_x+V_xB_y-V_yB_x)\right]\right\}\\
&+B_y\left\{\frac{\chi^M}{\mu_0(1+\chi^M)}B_y+\varepsilon_0\chi^E\left[V_x(E_z+V_xB_y-V_yB_x)-V_z(E_x+V_yB_z-V_zB_y)\right]\right\}\\
&+B_z\left\{\frac{\chi^M}{\mu_0(1+\chi^M)}B_z+\varepsilon_0\chi^E\left[V_y(E_x+V_yB_z-V_zB_y)-V_x(E_y+V_zB_x-V_xB_z)\right]\right\}
\end{aligned} \quad \textbf{(41)}
$$

$$I_x = V_x(-p + \tau_{xx} + \tau_{xx}^{EM}) + V_y(\tau_{xy} + \tau_{xy}^{EM}) + V_z(\tau_{xz} + \tau_{xz}^{EM})$$
$$I_y = V_x(\tau_{xy} + \tau_{xy}^{EM}) + V_y(-p + \tau_{yy} + \tau_{yy}^{EM}) + V_z(\tau_{yz} + \tau_{yz}^{EM})$$
$$I_z = V_x(\tau_{xz} + \tau_{xz}^{EM}) + V_y(\tau_{yz} + \tau_{yz}^{EM}) + V_z(-p + \tau_{zz} + \tau_{zz}^{EM})$$

(42)

$$H_x = k_1 B_x + k_2 V_y(E_z + V_x B_y - V_y B_x) - k_2 V_z(E_y + V_z B_x - V_x B_z)$$
$$H_y = k_1 B_y + k_2 V_z(E_x + V_y B_z - V_z B_y) - k_2 V_x(E_z + V_x B_y - V_y B_x)$$
$$H_z = k_1 B_z + k_2 V_x(E_y + V_z B_x - V_x B_z) - k_2 V_y(E_x + V_y B_z - V_z B_y)$$

$$T_x = P_y B_z - P_z B_y \; ; \; T_y = P_z B_x - P_x B_z ; \; T_z = P_x B_y - B_x P_y$$

(43)

Here

$$k_1 = \frac{1}{\mu_0 \left(1 + \chi^M\right)} \quad ; \quad k_2 = \varepsilon_0 \chi^E \qquad \textbf{(44)}$$

The entire system has twelve coupled partial differential equations and twelve unknowns given in the Eq. (35). This system can be routinely transformed into a general curvilinear nonorthogonal (ξ, η, ζ) system that is often used in direct finite volume discretization (Shankar, Hall and Alireza 1989; Shang 1991).

CONCLUSIONS

A system of twelve coupled partial differential equations governing three-dimensional unsteady flows of incompressible homocompositional electrically conducting polarizable and magnetizable fluids carrying electrically charged particles while exposed to unsteady externally applied nonuniform electric and magnetic fields has been formulated in its fully conservative vector operator and Cartesian coordinates forms. The fully conservative forms are suitable for either further transformations into other orthogonal and nonorthogonal coordinate systems or for direct discretization using difference and finite volume techniques.

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