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edited by
DENNIS A. SIGINER
AUBURN UNIVERSITY

G.S. DULIKRAVICH
PENNSYLVANIA STATE UNIVERSITY

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# ELECTRO-MAGNETO-HYDRODYNAMICS: (PART 2) A SURVEY OF MATHEMATICAL MODELS

**George S. Dulikravich**
Department of Aerospace Engineering
The Pennsylvania State University
State College, Pennsyslvania

**Sean R. Lynn**
Department of Aerospace Engineering
The Pennsylvania State University
State College, Pennsyslvania

## ABSTRACT

Fluid flow influenced by electric and magnetic fields has classically been divided into two separate fields of study: electro-hydrodynamics (EHD) studying flows under the influence of an electric field with free electric charges and no magnetic field, and magneto-hydrodynamics (MHD) studying flows under the influence of a magnetic field and no free electric charges or electric fields. This division was necessary to reduce the extreme complexity of the coupled system of Navier-Stokes, Maxwell's and constitutive equations describing combined electro-magneto-hydrodynamic (EMHD) flows. Recent advances in numerical techniques and computing power, as well as fully rigorous theoretical treatments, have made analysis of combined EMHD flows well within reach. The combined EMHD theory is compared with separate EHD and MHD models. This reveals the inconsistencies and shortcomings of classical formulations and allows discussion of the relative importance of terms describing the electro-magnetic force, electric current and heat transfer.

## NOMENCLATURE

| | | |
|---|---|---|
| $b$ | = | electric charge mobility coefficient, kg A $s^{-2}$ |
| $\mathbf{B}$ | = | magnetic field vector, kg $A^{-1}$ $s^{-2}$ |
| $\underline{\underline{\mathbf{d}}}$ | = | rate of deformation tensor, $s^{-1}$ |
| $\mathbf{D}$ | = | electric displacement field vector, A s $m^{-2}$ |
| $e$ | = | internal energy per unit mass, $m^2$ $s^{-2}$ |
| $\mathbf{E}$ | = | electric field vector, kg m $s^{-3}$ $A^{-1}$ |
| $\mathcal{E} = \mathbf{E}+\mathbf{v}\times\mathbf{B}$ | = | electromotive intensity vector, kg m $s^{-3}$ $A^{-1}$ |
| $\mathbf{f}$ | = | mechanical body force vector per unit mass, m $s^{-2}$ |
| $\mathbf{F}^{EM}$ | = | electromagnetic body force vector per unit volume, kg $m^{-2}$ $s^{-2}$ |
| $h$ | = | heat source or sink per unit mass, $m^2$ $s^{-3}$ |
| $\mathbf{H}$ | = | magnetic field strength vector, A $m^{-1}$ |
| $\mathbf{J}_\upsilon$ | = | apparent magnetization current vector, A $m^{-2}$ |
| $\mathbf{J}_d$ | = | electric drift current vector, A $m^{-2}$ |
| $\mathbf{J}_p$ | = | polarization current vector, A $m^{-2}$ |
| $\mathbf{J} = \mathbf{J}_d + \mathcal{J}$ | = | electric current vector, A $m^{-2}$ |
| $\mathcal{J}$ | = | electric conduction current vector, A $m^{-2}$ |
| $\kappa$ | = | thermal conductivity coefficient, kg m $s^{-3}$ $K^{-1}$ |
| $\mathbf{M}_\upsilon$ | = | magnetization vector per unit volume due to rotation of charged particles, A $m^{-1}$ |
| $\mathbf{M}_p$ | = | intrinsic or natural magnetization vector per unit volume, A $m^{-1}$ |
| $\mathbf{M}=\mathbf{M}_\upsilon+\mathbf{M}_p$ | = | total magnetization vector per unit volume, A $m^{-1}$ |
| $\mathcal{M}=\mathbf{M}+\mathbf{v}\times\mathbf{P}$ | = | magnetomotive intensity vector per unit volume, A $m^{-1}$ |
| $p$ | = | pressure, kg $m^{-1}$ $s^{-2}$ |
| $\mathbf{P}_e$ | = | polarization vector per unit volume due to electric charge, A s $m^{-2}$ |
| $\mathbf{P}_p$ | = | polarization vector per unit volume due to total dipole moments, A s $m^{-2}$ |
| $\mathbf{P} = \mathbf{P}_e+\mathbf{P}_p$ | = | total polarization vector per unit volume, A s $m^{-2}$ |
| $q_e$ | = | local free electric charge per unit volume, A s $m^{-3}$ |
| $q'_e=-\nabla\cdot\mathbf{P}_e$ | = | inhomogeneous electric charge per unit volume, A s $m^{-3}$ |
| $q_p=-\nabla\cdot\mathbf{P}_p$ | = | apparent electric charge per unit volume, A s $m^{-3}$ |
| $\dot{q}$ | = | heat flux vector, kg $s^{-3}$ |

| | | |
|---|---|---|
| $q_o = q_c - q'_c$ | = | free electric charge per unit volume, A s $m^{-3}$ |
| Q | = | point electric charge, A s |
| s | = | entropy per unit mass, $m^2\ kg^{-1}\ K^{-1}\ s^{-2}$ |
| $\underline{\underline{t}} = -p\underline{I}+\underline{\tau}$ | = | hydrodynamic stress tensor, kg $m^{-1}\ s^{-2}$ |
| v | = | fluid velocity vector, m $s^{-1}$ |
| V | = | absolute velocity vector, m $s^{-1}$ |
| $\mathcal{V}$ | = | volume, $m^3$ |

**Greek Symbols**

| | | |
|---|---|---|
| $\varepsilon$ | = | dielectric constant or electric permittivity coefficient, $kg^{-1}\ m^{-3}\ s^4\ A^2$ |
| $\varepsilon_o$ | = | dielectric constant or electric permittivity of vacuum ($\varepsilon_0 = 8.854 \times 10^{-12}$), $kg^{-1}\ m^{-3}\ s^4\ A^2$ |
| $\lambda$ | = | second viscosity coefficient, kg $m^{-1}\ s^{-1}$ |
| $\underline{\underline{\gamma}}$ | = | vorticity or spin tensor, $s^{-1}$ |
| $\sigma$ | = | electric conductivity coefficient, kg m $s^{-3}\ A^{-2}$ |
| $\eta$ | = | shear viscosity coefficient, kg $m^{-1}\ s^{-1}$ |
| $\theta$ | = | absolute temperature, K |
| $\rho$ | = | fluid density, kg $m^{-3}$ |
| $\tau$ | = | viscous stress tensor, kg $m^{-1}\ s^{-2}$ |
| $\mu$ | = | magnetic permeability coefficient, kg m $A^{-2}\ s^{-2}$ |
| $\mu_o$ | = | magnetic permeability of vacuum, ($\mu_0 = 4\pi \times 10^{-7}$), kg m $A^{-2}\ s^{-2}$ |
| $\chi^E$ | = | relative electric permitivitity or susceptibility ($\chi^E = \varepsilon/\varepsilon_o - 1$), non-dimensional |
| $\chi^M$ | = | relative magnetic permeability or susceptibility ($\chi^M = 1-\mu_o/\mu$), non-dimensional |
| $\Phi$ | = | viscous dissipation function, kg $m^{-1}s^{-3}$ |
| $\Psi$ | = | material free energy function, $m^2\ s^{-2}$ |

## 1. INTRODUCTION

The equations governing electro-magneto-hydrodynamic (EMHD) flows consist of the Navier-Stokes equations of fluid motion coupled with Maxwell's equations of electro-magnetics and material constitutive relations. The field has traditionally been divided into flows influenced only by electric fields and electric charges, and flows influenced only by magnetic fields and without electric charges. The former are called Electro-Hydrodynamic (EHD) flows and the latter Magneto-Hydrodynamic (MHD) flows (Stuetzer, 1962). Studies of EHD and MHD flows have ranged in complexity from the experimentally-based (Melcher, 1981) to more theoretically-based (Landau and Lifshitz, 1960). Much more recently, rigorous theoretical continuum mechanics treatments of EHD (Wineman and Rajagopal, 1995) and unified EMHD flows (Eringen and Maugin, 1990) have been developed. These continuum mechanics approaches are limited to non-relativistic, relatively low frequency phenomenon [Bergman, 1962; Lakhtakia, 1993].

Part 1 presented an overview of electro-magnetic theory with concentrated effort placed on the field-material interactions of polarization and magnetizaton. The unified EMHD theory (Eringen and Maugin, 1990) was also presented in Part 1, but more detailed discussion of this model is presented in Part 2 of this paper. Also presented in Part 2 is an overview of classical EHD and MHD models. The mainstay of this paper, however, is a comparison between classical EHD and MHD models and the unified EMHD theory. The comparison concentrates on similarities and differences between electro-magnetic force, electric current and heat conduction terms in the classical and unified models. Included in this is a discussion of the physical meaning and relative importance of classical model terms and recommendations for improving classical models. The inadequacies of simple superpositioning of classical models to fully describe unified EMHD flows are also noted.

## 2. GOVERNING SYSTEM OF EQUATIONS

The full system of equations governing unified EMHD flow consists of the Maxwell's equations governing electro-magnetism, the Navier-Stokes equations governing fluid flow and constitutive equations describing material behavior. This set of 27 equations allows solving for the 27 unknowns: fluid density ($\rho$), charge density ($q_o$), temperature ($\theta$), pressure (p), and the three vector components of velocity field (**v**), electric field (**E**), magnetic field (**B**), magnetization (**M**), polarization (**P**), current (**J**), heat flux ($\dot{\mathbf{q}}$), and conduction current ($\mathcal{J}$). The system of equations is made up by the set of equations 1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16 and an equation of state. Note that the conservation of electric charge has been grouped with Maxwell's equations, although it is not traditionally part of the system.

### 2.1 Maxwell Equations

Maxwell's equations are the system of linear differential equations governing electro-magnetic fields. They are given as (Eringen and Maugin, 1990, p. 504)

$$\nabla \cdot \mathbf{D} = q_o \tag{1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \tag{2}$$

or

$$\nabla \times \mathbf{B} = \mu_o\left(\nabla \times \mathbf{M} + \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{5}$$

also, the conservation of electric charges is given as

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{6}$$

Detailed descriptions of these equations can be found in any number of texts (Johnk, 1988; Cottingham and Greenwood, 1991; Haus and Melcher, 1989).

## 2.2 NAVIER-STOKES EQUATIONS

The equations of motion governing EMHD flow are the Navier-Stokes relations into which electromagnetic effects have been included. A summary of these equations is given below with a derivation from the global conservation law of continuum mechanics given in Appendix A. A rigorous derivation of these equations for electro-magnetic fluids is completed by Eringen and Maugin (1990).

**Conservation of mass**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{7}$$

**Conservation of momentum**

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot \left( \mathbf{v}\rho\mathbf{v} - \underline{\underline{\mathbf{t}}} \right) - \rho \mathbf{f} - \mathbf{F}^{EM} = 0 \tag{8}$$

where the electromagnetic force per unit volume is

$$\begin{aligned} \mathbf{F}^{EM} &= q_o \mathbf{E} + \mathbf{J} \times \mathbf{B} + (\nabla \mathbf{E}) \cdot \mathbf{P} + (\nabla \mathbf{B}) \cdot \mathbf{M} \\ &+ \nabla \cdot \left( (\mathbf{P} \times \mathbf{B}) \mathbf{v} \right) + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}) \end{aligned} \tag{9}$$

**Conservation of energy**

$$\begin{aligned} &\rho \frac{De}{Dt} + \frac{1}{2} \frac{\partial(\rho v^2)}{\partial t} + \nabla \cdot \left( \mathbf{v} \frac{1}{2} \rho v^2 \right) - \rho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} \\ &- \nabla \cdot \left( \underline{\underline{\mathbf{t}}} \cdot \mathbf{v} \right) + \left( \nabla \cdot \underline{\underline{\mathbf{t}}} \right) \cdot \mathbf{v} + \nabla \cdot \dot{\mathbf{q}} - \rho h \\ &- \rho \mathcal{E} \cdot \frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt} + \mathcal{M} \cdot \frac{D\mathbf{B}}{Dt} - \mathcal{J} \cdot \mathcal{E} = 0 \end{aligned} \tag{10}$$

**Clausius-Duheim Inequality**

$$\begin{aligned} \rho \frac{Ds}{Dt} &\geq \frac{\rho h + \Phi}{\theta} - \nabla \cdot \left( \frac{\dot{\mathbf{q}}}{\theta} \right) - \frac{\dot{\mathbf{q}} \cdot \nabla \theta}{\theta^2} \\ &+ \frac{\rho \mathcal{E} \cdot \frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt} - \mathcal{M} \cdot \frac{D\mathbf{B}}{Dt} + \mathcal{J} \cdot \mathcal{E}}{\theta} \end{aligned} \tag{11}$$

Here it is important to note that the same relationship can be obtained by considering the second law of thermodynamics and including viscous effects.

## 2.3 CONSTITUTIVE RELATIONS

The fluid (material) constitutive equations complete the system of equations governing EMHD flows and are given below. A fully rigorous derivation of these relations is given by Eringen and Maugin (1990). The relations presented in Part 1 are general and hold for non-isotropic fluids displaying nonlinear free energy characteristics. Assumptions of material isotropy and linearity simplify these equations considerably. As noted in Part 1, many materials display linear electro-magnetic characteristics and since most classical models assume material linearity, this assumption will be used throughout. As in Part 1 of this paper, it must be noted that the electro-magnetic material properties of the fluid may be dependent on physical properties of the flow, especially electro-magnetic frequency and temperature. The linear constitutive relations equations describing quasi-static or relatively low frequency polarization, magnetization and free energy characteristics of the fluid were discussed in Part 1, but reprinted here as

$$\mathbf{P} = -2\rho \frac{\partial \Psi}{\partial I_1} \mathcal{E} = \varepsilon_o \chi^E \mathcal{E} \tag{12}$$

$$\mathcal{M} = -2\rho \frac{\partial \Psi}{\partial I_2} \mathbf{B} = \frac{\chi^M}{\mu_o} \mathbf{B} \tag{13}$$

$$\Psi = \Psi_o - \frac{1}{2\rho} \left( \varepsilon_o \chi^E I_1 + \frac{\chi^M}{\mu_o} I_2 \right) \tag{14}$$

Most classical models consider only isotropic material and that assumption will be made here as well. Under the isotropic assumption all tensor quantities in the electric current and heat conduction relations drop out as well as all cross-product terms. The reason for removal of cross-product and tensor quantities in the linear, isotropic constitutive relations is that the order of the integrity basis for the governing constitutive relations has been lowered (Eringen and Maugin, 1990, pp. 154, 173). The phenomenological current given by the constitutive theory is (Eringen and Maugin, 1990, p. 162)

$$\begin{aligned} \mathcal{J} &= \left( \sigma_1 + \sigma_3 \underline{\underline{\mathbf{d}}} + \sigma_7 \underline{\underline{\gamma}} \right) \mathcal{E} + \left( \sigma_2 + \sigma_4 \underline{\underline{\mathbf{d}}} + \sigma_8 \underline{\underline{\gamma}} \right) \nabla \theta \\ &+ \left[ \sigma_5 \mathcal{E} \times \mathbf{B} + \sigma_{11} \left( \underline{\underline{\mathbf{d}}} (\mathcal{E} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \mathcal{E}) \times \mathbf{B} \right) \right] \\ &+ \left[ \sigma_6 \nabla \theta \times \mathbf{B} + \sigma_{12} \left( \underline{\underline{\mathbf{d}}} (\nabla \theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \nabla \theta) \times \mathbf{B} \right) \right] \\ &+ \sigma_9 (\mathbf{B} \cdot \mathcal{E}) \mathbf{B} + \sigma_{10} (\mathbf{B} \cdot \nabla \theta) \mathbf{B} \end{aligned} \tag{15}$$

This current is called the conduction current and it will be discussed in more detail in a latter section. Similarly the constitutive relation for heat flux is given as (Eringen and Maugin, 1990, p. 161)

$$\dot{\mathbf{q}} = \left(\kappa_2 + \kappa_4 \underline{\underline{\mathbf{d}}} + \kappa_8 \underline{\underline{\gamma}}\right)\mathcal{E} + \left(\kappa_1 + \kappa_3 \underline{\underline{\mathbf{d}}} + \kappa_7 \underline{\underline{\gamma}}\right)\nabla\theta$$
$$+ \left[\kappa_6 \mathcal{E} \times \mathbf{B} + \kappa_{12}\left(\underline{\underline{\mathbf{d}}}(\mathcal{E} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\mathcal{E}) \times \mathbf{B}\right)\right]$$
$$+ \left[\kappa_5 \nabla\theta \times \mathbf{B} + \kappa_{11}\left(\underline{\underline{\mathbf{d}}}(\nabla\theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\nabla\theta) \times \mathbf{B}\right)\right]$$
$$+ \kappa_9(\mathbf{B} \cdot \nabla\theta)\mathbf{B} + \kappa_{10}(\mathbf{B} \cdot \mathcal{E})\mathbf{B} \quad \textbf{(16)}$$

In the above relations, the tensor $\underline{\underline{\mathbf{d}}}$ is the symmetric part of the velocity gradient tensor. It is called the rate of deformation tensor and given as (Eringen and Maugin, 1990, p. 13)

$$\underline{\underline{\mathbf{d}}} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \quad \textbf{(17)}$$

Similarly, the tensor $\underline{\underline{\gamma}}$ is the anti-symmetric part of the velocity gradient tensor. It is called the vorticity or spin tensor given as (Eringen and Maugin, 1990, p. 13)

$$\underline{\underline{\gamma}} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \quad \textbf{(18)}$$

It can be seen from equations 15 and 16 that the electromagnetic field is not the only cause of electric current and the temperature gradient is not the only source of heat conduction as is commonly assumed. The electric field, magnetic field and heat conduction may couple to produce charge motion and heat transfer. These couplings are called phenomenological cross effects and may be placed in four general categories: 1) thermoelectric, 2) galvanomagnetic, 3) thermomagnetic, and 4) second order effects (Eringen and Maugin, 1990, pp. 161-163). These categories are based on the source of the effect and each will be described in turn.

**2.3.1 Thermoelectric Effects**

Thermoelectric effects are caused by couplings between the temperature gradient and the electric field. Reducing the current and heat conduction equations to their thermoelectric terms yields

$$\mathcal{J} = \left(\sigma_1 + \sigma_3 \underline{\underline{\mathbf{d}}} + \sigma_7 \underline{\underline{\gamma}}\right)\mathcal{E} + \left(\sigma_2 + \sigma_4 \underline{\underline{\mathbf{d}}} + \sigma_8 \underline{\underline{\gamma}}\right)\nabla\theta$$
$$\dot{\mathbf{q}} = \left(\kappa_2 + \kappa_4 \underline{\underline{\mathbf{d}}} + \kappa_8 \underline{\underline{\gamma}}\right)\mathcal{E} + \left(\kappa_1 + \kappa_3 \underline{\underline{\mathbf{d}}} + \kappa_7 \underline{\underline{\gamma}}\right)\nabla\theta \quad \textbf{(19)}$$

From these equations it can readily be seen that a temperature gradient in the material produces an electric current and that an applied electric field produces heat transfer in the material. A temperature gradient producing an electric current is referred to as the Thompson effect while an electric field producing heat conduction is termed the Peltier effect. These two effects together are the Seebeck effect and form the basis for thermocouples. Also note that the first term in the electric current and the second term in the heat conduction equations are not true cross effects; they are the Ohmic charge conduction and Fourier heat transfer respectively.

**2.3.2 Galvanometric Effects**

When the electric and magnetic fields are not parallel, electric current and heat conduction are induced in the material. These set of effects are termed galvanomagnetic. Reducing the full current and heat conduction equations to their galvanomagnetic effects yields

$$\mathcal{J} = \sigma_5 \mathcal{E} \times \mathbf{B} + \sigma_{11}\left(\underline{\underline{\mathbf{d}}}(\mathcal{E} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\mathcal{E}) \times \mathbf{B}\right)$$
$$\dot{\mathbf{q}} = \kappa_6 \mathcal{E} \times \mathbf{B} + \kappa_{12}\left(\underline{\underline{\mathbf{d}}}(\mathcal{E} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\mathcal{E}) \times \mathbf{B}\right) \quad \textbf{(20)}$$

Electric current induction from non-parallel electric and magnetic field is called the Hall effect. In analogy, heat conduction produced by non-parallel electric and magnetic fields is called the Ettingshausen effect (Eringen and Maugin, 1990, pp. 161-163).

**2.3.3 Thermomagnetic Effects**

When the temperature gradient and the magnetic field are not parallel, electric current (Nernst effect) and heat conduction (Righi-LeDuc effect) are induced in the material. These effects are termed thermomagnetic. Reducing the full current and heat conduction equations to their thermomagnetic effects yields

$$\mathcal{J} = \sigma_6 \nabla\theta \times \mathbf{B} + \sigma_{12}\left(\underline{\underline{\mathbf{d}}}(\nabla\theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\nabla\theta) \times \mathbf{B}\right)$$
$$\dot{\mathbf{q}} = \kappa_5 \nabla\theta \times \mathbf{B} + \kappa_{11}\left(\underline{\underline{\mathbf{d}}}(\nabla\theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}}\nabla\theta) \times \mathbf{B}\right) \quad \textbf{(21)}$$

**2.3.4 Secondary Effects**

Finally, secondary effects are caused by non-orthogonality between the electric field and magnetic field and the temperature gradient and magnetic field. The electric current and heat conduction equations with only secondary effects are shown below.

$$\mathcal{J} = \sigma_9(\mathbf{B} \cdot \mathcal{E})\mathbf{B} + \sigma_{10}(\mathbf{B} \cdot \nabla\theta)\mathbf{B}$$
$$\dot{\mathbf{q}} = \kappa_{10}(\mathbf{B} \cdot \mathcal{E})\mathbf{B} + \kappa_9(\mathbf{B} \cdot \nabla\theta)\mathbf{B} \quad \textbf{(22)}$$

## 3. CLASSICAL ELECTRO-HYDRODYNAMICS

As mentioned previously EHD flows are those in which magnetic effects may be neglected and charged particles are present. One of the implied assumptions for this paper, is that the flows are at non-relativistic speeds. There are cases, particularly in astrophysical MHD, where this assumption cannot be made (Hughes and Young, 1966). The other major assumption made in classical EHD is that only a quasi-static electric field is applied so that the magnetic field, both applied and induced, may be neglected. Atten and Moreau (1972) present a detailed coverage of classical EHD modeling and discuss the relative importance of terms in the force and electric current through stability analysis. In order for this assumption to apply the fluid must contain electrically charged particles. With these assumptions, three Maxwell's equations govern the flow (Stuetzer, 1966).

$$\nabla \cdot \mathbf{D} = q_o \tag{23}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{24}$$

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{25}$$

Modifications to the Navier-Stokes relations come from the electro-magnetic force on the fluid from which all magnetic field terms have been neglected. Otherwise the equations appear the same as in section 2.2. With classical EHD assumptions the electro-magnetic force in the unified EMHD theory becomes :

$$\mathbf{F}^{EM} = q_o\mathbf{E} + (\nabla \mathbf{E}) \cdot \mathbf{P} \tag{26}$$

This is not the form of the electro-magnetic force usually seen in classical EHD formulations (Stuetzer, 1962). Through the use of thermodynamics, equation 26 and the material constitutive equation of state, the electro-magnetic force is usually derived in the following equivalent forms (Eringen and Maugin, 1990, pp. 505-507; Landau and Lifshitz, 1960, pp. 59-63)

$$\mathbf{F}^{EM} = q_o\mathbf{E} - \frac{\mathbf{E}^2}{2}\nabla\varepsilon + \frac{1}{2}\nabla\left(\mathbf{E}^2\rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_\theta\right) \tag{27}$$

or as

$$\mathbf{F}^{EM} = q_o\mathbf{E} - \frac{\mathbf{E}^2}{2}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_\rho \nabla\theta + \frac{1}{2}\nabla\left(\mathbf{E}^2\rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_\theta\right) \tag{28}$$

Equation 28 is most common electro-magnetic force formulation in classical EHD. The three terms in the equation are the electrophoretic, dielectrophoretic and electrostrictive terms, respectively. These terms require more explanation.

The *electrophoretic force* or Coulomb force is caused by the electric field acting on free charges in the fluid. It is an irrotational force except when charge gradients are present (Poulter and Allen, 1986). The *dielectrophoretic force* is also a translational force, but is caused by polarization of the fluid and/or particles in the fluid. A dielectrophoretic force will occur where high gradients of electric permittivity are present. This condition will be true in high temperature gradient flows, multi-constituent flows, particulate flows (Pohl 1978) or any time the electric field must pass through two contacting media of different permitivities (Aoyama et al. 1993). Grassi and DiMarco (1991) treat the dielectrophoretic force as it applies to bubbly flows and heat transfer. Poulter and Allen (1986) note that the dielectrophoretic force produces greatest circulation when the dielectric permitivity is inhomogeneous and non-parallel with the applied electric field. The last force, the *electrostrictive force*, is a distortive force (as opposed to the previous translational forces) associated with fluid compression and shear. The electrostrictive force is usually smaller than the -phoretic forces, but is present in high pressure gradient flows, compressible flows and flows with a non-uniform applied electric field. In hydrodynamically bounded systems, the electrostrictive force plays no part due to its irrotational nature (Poulter and Allen, 1986). Pohl (1978) describes this phenomenon in greater detail.

Classical EHD modeling derives directly from the unified EMHD theory. The same can be said of the constitutive current relation. From the unified EMHD theory the electric current, assuming material isotropy and linearity, is given by the relation

$$\mathbf{J} = q_o\mathbf{v} + \sigma_1\mathcal{E} + \sigma_2\nabla\theta \tag{29}$$

This is not the form seen in classical EHD models however (Stuetzer 1962). Classical EHD modeling typically defines the electric current as simply the first two terms of equation 29: the convective and conductive current, respectively. However, more advanced classical models define the current as (Eringen and Maugin, 1990, p. 562)

$$\mathbf{J} = q_o\mathbf{v} + q_o b\mathbf{E} - \mathcal{D}\nabla q_o \tag{30}$$

At first glance equation 29 seems not to match equation 30, or, seems to imply that the temperature gradient is directly related to the electric charge gradient. This may be shown to be true as the last two terms in equation 30 come from the Einstein-Fokker relationships, derived from studies of Brownian motion (de Groot and Mazur, 1962, p. 264-273), which relate any concentration gradient to a mobility ($q_o b\mathbf{E}$) and a diffusion ($\mathcal{D}\nabla q_o$). A more heuristic proof of this relation can be obtained by expanding the material equation of state, which by linear theory relates pressure, temperature and density, in a Taylor series around density and pressure. Then, because the electric current is desired, we consider only electric field and charge contributions to the pressure. Thus, the gradient of velocity becomes an equivalent electric charge gradient and the gradient of pressure becomes an equivalent of the electrophoretic force (both multiplied by constants). By performing a unit analysis on the constants, it can be seen that the constant multiplied by the pressure gradient is equal to the charge mobility coefficient, b, and the constant multiplied by the charge gradient is equal to the charge diffusion coefficient, $\mathcal{D}$. By either de Groot and Mazur's rigorous non-equilibrium thermodynamics method or the method described above, equation 29 may be shown to be equivalent to equation 30. Newman (1991) also provides a detailed discussion of the concepts of diffusion and mobility. The second, or diffusive term, is often neglected where limited free charges are available (Schilling and Schachter, 1967).

The final equation describing classical EHD flow is the heat transfer constitutive relation. This is the one area where classical EHD theory does not match the unified EMHD theory with EHD assumptions. By introducing classical EHD assumptions in the unified EMHD theory the following relationship is obtained for heat transfer

$$\dot{\mathbf{q}} = \kappa_2\mathbf{E} + \kappa_1\nabla\theta \tag{31}$$

This relationship, however is not the one commonly seen in classical EHD models. These models usually neglect the contribution to heat transfer from the electric field so that equation 31 becomes

$$\dot{\mathbf{q}} = \kappa \nabla \theta \tag{32}$$

This relation is Fourier's law of heat conduction. Here, it must be noted that although classical EHD modeling seems to neglects heat transfer induced by the electric field and electric current, Joule heating ($\mathbf{J} \cdot \mathbf{E}$) from equation 11, is usually included in the EHD analysis.

## 4. CLASSICAL MAGNETO-HYDRODYNAMICS

The classical modeling of magneto-hydrodynamics makes non-relativistic assumption just as in classical EHD theory. However, where classical EHD made the assumption of quasi-electrostatics, classical MHD theory makes a quasi-magnetostatic assumption. Thus, the direct electric field induces a magnetic field of much less magnitude than the applied magnetic field. This assumption also implies that electric current comes primarily from conductive means and that there are no free charges in the fluid. With these assumptions Maxwell's equations become (Stuetzer, 1966)

$$\nabla \cdot \mathbf{B} = 0 \tag{33}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{34}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{35}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{36}$$

The modifications to the Navier-Stokes relations come from the electro-magnetic force on the fluid from which all induced electric field terms have been neglected. Otherwise, the equations appear the same as in section 2.2. With classical MHD assumptions the electric force in the unified EMHD theory becomes (Stuetzer, 1962)

$$\mathbf{F}^{EM} = \mathbf{J} \times \mathbf{B} + (\nabla \mathbf{B}) \cdot \mathbf{M} \tag{37}$$

The second term, source of dimagnetophoretic and magnetostrictive forces is typically neglected in MHD. Thus, the electro-magnetic force for MHD becomes (Stuetzer, 1962)

$$\mathbf{F}^{EM} = \mathbf{J} \times \mathbf{B} \tag{38}$$

An improvement in classical MHD modeling could be made by including the dimagnetophoretic and magnetostrictive terms, especially in cases where MHD flow conditions satisfy conditions analogous to cases where dielectrophoretic and electrostrictive effects are important in EHD (see section 3).

By making classical MHD assumptions the unified EMHD current becomes

$$\begin{aligned}\mathbf{J} &= \sigma_1 \mathcal{E} + \sigma_2 \nabla\theta + \sigma_5 \mathcal{E} \times \mathbf{B} + \sigma_6 \nabla\theta \times \mathbf{B} \\ &\quad + \sigma_9 (\mathbf{B} \cdot \mathcal{E})\mathbf{B} + \sigma_{10} (\mathbf{B} \cdot \nabla\theta)\mathbf{B}\end{aligned} \tag{39}$$

However, classical theory usually defines the current as (Eringen and Maugin, 1990, p. 510)

$$\mathbf{J} = \sigma \mathbf{E} + \sigma(\mathbf{v} \times \mathbf{B}) + \sigma^{\theta} \nabla\theta = \sigma \mathcal{E} + \sigma^{\theta} \nabla\theta \tag{40}$$

Here $\sigma^{\theta}$ is the Seebeck coefficient (Eringen and Maugin, 1990, p. 174). It is seen to be equivalent to $\sigma_2$ in equation 39. Note that in some classical MHD formulations the Seebeck coefficient is not used (Stuetzer, 1962). Regardless, the classical MHD formulation neglects a significant number of effects. Improvements could be made to the classical theory by including terms from equation 39 depending on the details of the flow problem in question.

The final relation of comparison between the unified EMHD model and the classical MHD model is the heat transfer. Once again in classical modeling, Joule heating is often included in the energy relation, but the heat transfer constitutive relation remains the same as in equation 11. In comparison the unified EMHD model with classical MHD assumptions is

$$\begin{aligned}\dot{\mathbf{q}} &= \kappa_2 \mathcal{E} + \kappa_1 \nabla\theta + \kappa_6 \mathcal{E} \times \mathbf{B} + \kappa_5 \nabla\theta \times \mathbf{B} \\ &\quad + \kappa_9 (\mathbf{B} \cdot \nabla\theta)\mathbf{B} + \kappa_{10} (\mathbf{B} \cdot \mathcal{E})\mathbf{B}\end{aligned} \tag{41}$$

As can be seen, the classical MHD modeling neglects many effects. Improvements could be made by including the -strictive effects of equation 37 as well as cross-effects between electric current and heat transfer.

Whereas the classical EHD model includes many of important effects and matches the unified EMHD theory very well, classical MHD formulations could be improved. Depending on the problem being studied, improvements in the force, current and heat transfer terms could be made. As in classical EHD modeling, it is important to be aware of the fact that many force, current and heat transfer terms can be written in several different formats, each of which is equivalent. It is therefore important to recognize the potential danger in simply adding terms from different MHD models.

## 5. CONCLUSION

The objective of this paper was to survey sufficient background material to allow initial implementation of a unified EMHD theory presented by Eringen and Maugin. To accomplish this the basics of electro-magnetic field theory was presented in Part 1, with emphasis placed on describing the causes and effects of material polarization and magnetization. This paper presented the equations governing unified EMHD flows. The sixteen equations governing flow characteristics are contained in the Maxwell's equations of electro-magnetic fields, the Navierequations governing flow characteristics are contained in the Maxwell's equations of electro-magnetic fields, the Navier-Stokes flow-field equations and the constitutive relations for current, heat transfer and material equation of state.

The heart of the paper is a presentation of classical models for EHD and MHD flows compared with the unified EMHD theory. Both classical models assumed material isotropy and linear constitutive theory which was shown often to be a valid assumption (Eringen and Maugin, 1990). It was shown that classical EHD models matched nearly identically with the EMHD theory. The classical equations for heat transfer were the only place where EHD and EMHD models did not match. Finally, it was shown that even though the EHD and EMHD models matched, they were often written in different forms, making them seem incompatible at first glance.

A similar comparison between the classical MHD model and the unified EMHD model was performed. Unlike the comparison with EHD, the unified model did not compare well with the classical model. Dimagnetophoretic and magnetostrictive terms were not included in classical MHD modeling of force. Being defined as first order effects (Eringen and Maugin, 1990) these terms should be considered depending on the characteristics of the particular flow being analyized. Further, classical MHD theory was shown to neglect several cross-effect terms in the formulation of the electric current. Classical MHD theory did, however include all first order effects in the electric current formulation.

Finally, because many of the terms in electro-magnetic force, electric current and heat transfer may be written in different forms, it is important to recognize the danger of simply superpositioning of terms from classical models without fully understanding their meanings.

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## APPENDIX A

### THE GLOBAL CONSERVATION LAW

The Navier-Stokes equations of fluid motion can be derived from the global conservation law given as (Eringen and Maugin, 1990, p. 68)

$$\frac{D}{Dt}\int_{\mathcal{V}-\sigma} \phi \, d\mathcal{V} + \frac{\delta}{\delta t}\int_{\sigma} \Phi da = \int_{\partial\mathcal{V}-\sigma} \mathcal{T} \cdot \mathbf{n} \, da + \int_{\mathcal{V}-\sigma} (\mathbf{g}+\mathbf{g}^{E}) \, d\mathcal{V} + \int_{\sigma} [\hat{\mathbf{g}}] \cdot \mathbf{n} \, da \qquad \text{(A.1)}$$

where

$$\begin{aligned}
\frac{\delta}{\delta t} &= \text{time rate following the surface} \\
\sigma &= \text{discontinuity surface} \\
\varphi &= \text{surface density of } \phi \text{ on } \sigma \\
\mathcal{T} &= \text{flux of } \phi \\
\mathbf{g} &= \text{body forces} \\
\mathbf{g}^{EM} &= \text{electromagnetic body forces} \\
\mathbf{v} &= \text{fluid velocity} \\
\mathbf{V} &= \text{absolute velocity}
\end{aligned} \qquad \text{(A.2)}$$

This law will be manipulated to achieve a differential form of the governing equations. For no surface density, $\varphi=0$, use of transport theorem and the Green-Gauss equation results in

$$\int_{\mathcal{V}-\sigma}\frac{\partial\phi}{\partial t} d\mathcal{V} + \int_{\partial\mathcal{V}-\sigma}\phi\mathbf{v}\cdot d\mathbf{a} - \int_{\sigma(t)}[\phi\mathbf{n}]\cdot d\mathbf{a} = \int_{\partial\mathcal{V}-\sigma}\mathcal{T}\cdot\mathbf{n} \, da + \int_{\mathcal{V}-\sigma}(\mathbf{g}+\mathbf{g}^{E})d\mathcal{V} + \int_{\sigma}[\hat{\mathbf{g}}]\cdot\mathbf{n} \, da \qquad \text{(A.3)}$$

Through the use of the transport theorem (Eringen and Maugin, 1990, p. 20)

$$\frac{D}{Dt}\int_{\mathcal{V}-\sigma}\phi \, d\mathcal{V} = \int_{\mathcal{V}-\sigma}\frac{\partial\phi}{\partial t}d\mathcal{V} + \int_{\partial\mathcal{V}-\sigma}\phi\mathbf{v}\cdot d\mathbf{a} - \int_{\sigma(t)}[\phi\mathbf{v}]\cdot d\mathbf{a} \qquad \text{(A.4)}$$

and by using the Green-Gauss theorem (Eringen and Maugin, 1990, p. 20)

$$\int_{\mathcal{V}-\sigma}\nabla\cdot\mathbf{A} \, d\mathcal{V} + \int_{\sigma(t)}[\mathbf{A}]\cdot d\mathbf{a} = \int_{\partial\mathcal{V}-\sigma}\mathbf{A}\cdot d\mathbf{a} \qquad \text{(A.5)}$$

Further algebraic manipulation results in

$$\int_{\mathcal{V}-\sigma}\left(\frac{\partial\phi}{\partial t}+\nabla\cdot(\phi\mathbf{v})\right)d\mathcal{V} + \int_{\sigma(t)}[\phi(\mathbf{v}-\mathbf{V})]\cdot d\mathbf{a} = \int_{\partial\mathcal{V}-\sigma}\mathcal{T}\cdot\mathbf{n} \, da + \int_{\mathcal{V}-\sigma}(\mathbf{g}+\mathbf{g}^{E})d\mathcal{V} + \int_{\sigma}[\hat{\mathbf{g}}]\cdot\mathbf{n} \, da \qquad \text{(A.6)}$$

and

$$\int_{\mathcal{V}-\sigma}\left(\frac{\partial\phi}{\partial t}+\nabla\cdot(\phi\mathbf{v})\right)d\mathcal{V} + \int_{\sigma(t)}[\phi(\mathbf{v}-\mathbf{V})]\cdot d\mathbf{a} = \int_{\mathcal{V}-\sigma}\nabla\cdot\tau \, d\mathcal{V} + \int_{\sigma(t)}[\tau]\cdot d\mathbf{a} + \int_{\mathcal{V}-\sigma}(\mathbf{g}+\mathbf{g}^{E})d\mathcal{V} + \int_{\sigma}[\hat{\mathbf{g}}]\cdot\mathbf{n} \, da \qquad \text{(A.7)}$$

and

$$\int_{\mathcal{V}-\sigma}\left(\frac{\partial\phi}{\partial t}+\nabla\cdot(\mathbf{v}\phi-\tau)-\mathbf{g}-\mathbf{g}^{E}\right)d\mathcal{V} + \int_{\sigma}[\phi(\mathbf{v}-\mathbf{V})-\tau-\hat{\mathbf{g}}^{E}]\cdot\mathbf{n} \, da = 0 \qquad \text{(A.8)}$$

Thus the general expression for a conservation law in the domain, $\mathcal{V}$, and on the surface, $\sigma$, is (Eringen and Maugin, 1990, p. 76)

$$\frac{\partial\phi}{\partial t}+\nabla\cdot(\mathbf{v}\phi-\mathcal{T})-\mathbf{g}-\mathbf{g}^{E}=0 \qquad \text{(A.9)}$$

With this derived, the mass, momentum and energy conservation laws can now be derived.

### MASS CONSERVATION

The equation of mass conservation can be derived by applying the following parameters to equation A.9 as

$$\begin{aligned}
\phi &= \rho \\
\mathcal{T} &= 0 \\
\mathbf{g} &= 0 \\
\mathbf{g}^{E} &= 0
\end{aligned} \qquad \text{(A.10)}$$

resulting in

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad \text{(A.11)}$$

**MOMENTUM CONSERVATION**

The linear momentum conservation can be obtained with the following parameters substituted into equation A.9 (Eringen and Maugin, 1990, p. 77)

$$\begin{aligned}\phi &= \rho \mathbf{v}\\ \mathcal{T} &= \underline{\underline{\mathbf{t}}}\\ \mathbf{g} &= \rho \mathbf{f}\\ \mathbf{g}^E &= \mathbf{F}^{EM}\end{aligned} \tag{A.12}$$

where the electromagnetic force vector per unit volume is defined as (Eringen and Maugin, 1990, p. 505)

$$\begin{aligned}\mathbf{F}^{EM} &= q_o\mathbf{E} + \mathbf{J}\times\mathbf{B} + (\nabla\mathbf{E})\cdot\mathbf{P} + (\nabla\mathbf{B})\cdot\mathbf{M}\\ &\quad + \nabla\cdot\left((\mathbf{P}\times\mathbf{B})\mathbf{v}\right) + \frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B})\end{aligned} \tag{A.13}$$

Consequently, the conservation of momentum for unified EMHD flows is

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\left(\mathbf{v}\rho\mathbf{v} - \underline{\underline{\mathbf{t}}}\right) - \rho\mathbf{f} - \mathbf{F}^{EM} = 0 \tag{A.14}$$

**ENERGY CONSERVATION**

Similarly, the conservation of energy equation can be obtained with the following parameters substituted into equation A.9 (Eringen and Maugin, 1990, p. 78)

$$\begin{aligned}\phi &= \rho e + \frac{1}{2}\rho v^2\\ \mathcal{T} &= \underline{\underline{\mathbf{t}}}\cdot\mathbf{v} - \dot{\mathbf{q}}\\ \mathbf{g} &= \rho\mathbf{f}\cdot\mathbf{v} + \rho\mathbf{h}\\ \mathbf{g}^E &= \mathbf{W}^{EM}\end{aligned} \tag{A.15}$$

where the electromagnetic energy vector per unit volume is defined as (Eringen and Maugin, 1990, p. 62)

$$\begin{aligned}\mathbf{W}^{EM} &= \mathbf{F}^{EM}\cdot\mathbf{v} + \rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}\\ &\quad - \mathcal{M}\cdot\frac{D\mathbf{B}}{Dt} + \mathcal{J}\cdot\mathcal{E}\end{aligned} \tag{A.16}$$

The electric conduction current vector for unified EMHD effects is (Eringen and Maugin, 1990, p. 61)

$$\begin{aligned}\mathcal{J} &= \left(\sigma_1 + \sigma_3\underline{\underline{\mathbf{d}}} + \sigma_7\underline{\underline{\gamma}}\right)\mathcal{E} + \left(\sigma_2 + \sigma_4\underline{\underline{\mathbf{d}}} + \sigma_8\underline{\underline{\gamma}}\right)\nabla\theta\\ &\quad + \left[\sigma_5\mathcal{E}\times\mathbf{B} + \sigma_{11}\left(\underline{\underline{\mathbf{d}}}(\mathcal{E}\times\mathbf{B}) - (\underline{\underline{\mathbf{d}}}\mathcal{E})\times\mathbf{B}\right)\right]\\ &\quad + \left[\sigma_6\nabla\theta\times\mathbf{B} + \sigma_{12}\left(\underline{\underline{\mathbf{d}}}(\nabla\theta\times\mathbf{B}) - (\underline{\underline{\mathbf{d}}}\nabla\theta)\times\mathbf{B}\right)\right]\\ &\quad + \sigma_9(\mathbf{B}\cdot\mathcal{E})\mathbf{B} + \sigma_{10}(\mathbf{B}\cdot\nabla\theta)\mathbf{B}\end{aligned} \tag{A.17}$$

The heat flux vector for unified EMHD effects is (Eringen and Maugin, 1990, p. 61)

$$\begin{aligned}\dot{\mathbf{q}} &= \left(\kappa_2 + \kappa_4\underline{\underline{\mathbf{d}}} + \kappa_8\underline{\underline{\gamma}}\right)\mathcal{E} + \left(\kappa_1 + \kappa_3\underline{\underline{\mathbf{d}}} + \kappa_7\underline{\underline{\gamma}}\right)\nabla\theta\\ &\quad + \left[\kappa_6\mathcal{E}\times\mathbf{B} + \kappa_{12}\left(\underline{\underline{\mathbf{d}}}(\mathcal{E}\times\mathbf{B}) - (\underline{\underline{\mathbf{d}}}\mathcal{E})\times\mathbf{B}\right)\right]\\ &\quad + \left[\kappa_5\nabla\theta\times\mathbf{B} + \kappa_{11}\left(\underline{\underline{\mathbf{d}}}(\nabla\theta\times\mathbf{B}) - (\underline{\underline{\mathbf{d}}}\nabla\theta)\times\mathbf{B}\right)\right]\\ &\quad + \kappa_9(\mathbf{B}\cdot\nabla\theta)\mathbf{B} + \kappa_{10}(\mathbf{B}\cdot\mathcal{E})\mathbf{B}\end{aligned} \tag{A.18}$$

Then, from the general for of the differential conservation law, A.9, the energy conservation becomes

$$\begin{aligned}& e\frac{\partial\rho}{\partial t} + \rho\frac{\partial e}{\partial t} + \frac{1}{2}\frac{\partial\left(\rho v^2\right)}{\partial t}\\ & + \nabla\cdot\left(\mathbf{v}\left(\rho e + \frac{1}{2}\rho v^2\right)\right) - \nabla\cdot\left(\underline{\underline{\mathbf{t}}}\cdot\mathbf{v}\right)\\ & + \nabla\cdot\dot{\mathbf{q}} - \rho\mathbf{f}\cdot\mathbf{v} - \rho h - \mathbf{F}^{EM}\cdot\mathbf{v}\\ & - \rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt} + \mathcal{M}\cdot\frac{D\mathbf{B}}{Dt} - \mathcal{J}\cdot\mathcal{E} = 0\end{aligned} \tag{A.19}$$

From the conservation of linear momentum it follows that

$$-\mathbf{F}^{EM}\cdot\mathbf{v} = \left(\nabla\cdot\underline{\underline{\mathbf{t}}}\right)\cdot\mathbf{v} + \rho\mathbf{f}\cdot\mathbf{v} - \rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v} \tag{A.20}$$

so that the energy conservation equation becomes

$$
\begin{aligned}
&e\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})\right)+\rho\left(\frac{\partial e}{\partial t}+\mathbf{v}\cdot\nabla e\right)\\
&+\frac{1}{2}\frac{\partial(\rho v^2)}{\partial t}+\nabla\cdot\left(\mathbf{v}\frac{1}{2}\rho v^2\right)-\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v}\\
&-\nabla\cdot\left(\underline{\underline{\mathbf{t}}}\cdot\mathbf{v}\right)+\left(\nabla\cdot\underline{\underline{\mathbf{t}}}\right)\cdot\mathbf{v}+\nabla\cdot\dot{\mathbf{q}}-\rho h\\
&-\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}+\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}-\mathcal{J}\cdot\mathcal{E}=0
\end{aligned}
\qquad \text{(A.21)}
$$

Substituting the mass conservation relation from A.11, the energy conservation becomes

$$
\begin{aligned}
&\rho\frac{De}{Dt}+\frac{1}{2}\frac{\partial(\rho v^2)}{\partial t}+\nabla\cdot\left(\mathbf{v}\frac{1}{2}\rho v^2\right)\\
&-\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v}-\nabla\cdot\left(\underline{\underline{\mathbf{t}}}\cdot\mathbf{v}\right)+\left(\nabla\cdot\underline{\underline{\mathbf{t}}}\right)\cdot\mathbf{v}+\nabla\cdot\dot{\mathbf{q}}\\
&-\rho h-\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}+\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}-\mathcal{J}\cdot\mathcal{E}=0
\end{aligned}
\qquad \text{(A.22)}
$$

For simplicity let

$$
\begin{aligned}
C=&-\nabla\cdot\left(\underline{\underline{\mathbf{t}}}\cdot\mathbf{v}\right)+\left(\nabla\cdot\underline{\underline{\mathbf{t}}}\right)\cdot\mathbf{v}+\nabla\cdot\dot{\mathbf{q}}-\rho h\\
&-\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}+\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}-\mathcal{J}\cdot\mathcal{E}
\end{aligned}
\qquad \text{(A.23)}
$$

Then, the energy conservation becomes

$$
\begin{aligned}
&\rho\frac{De}{Dt}+\frac{v^2}{2}\left(\frac{\partial\rho}{\partial t}+(\nabla\cdot\rho\mathbf{v})\right)\\
&\frac{\rho}{2}\left(\frac{\partial v^2}{\partial t}+(\mathbf{v}\cdot\nabla)(v^2)\right)\\
&-\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v}+C=0
\end{aligned}
\qquad \text{(A.24)}
$$

Again, evoking mass conservation, this becomes

$$
\rho\frac{De}{Dt}+\frac{1}{2}\rho\frac{Dv^2}{Dt}-\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v}+C=0 \qquad \text{(A.25)}
$$

Since

$$
-\nabla\cdot(\underline{\underline{\mathbf{t}}}\cdot\mathbf{v})+(\nabla\cdot\underline{\underline{\mathbf{t}}})\cdot\mathbf{v}=-\underline{\underline{\mathbf{t}}}:\nabla\mathbf{v} \qquad \text{(A.26)}
$$

the energy conservation for unified EMHD flow becomes (Eringen and Maugin, 1990, p. 178)

$$
\begin{aligned}
&\rho\frac{De}{Dt}-\underline{\underline{\mathbf{t}}}:\nabla\mathbf{v}+\nabla\cdot\dot{\mathbf{q}}-\rho h-\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}\\
&+\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}-\mathcal{J}\cdot\mathcal{E}=0
\end{aligned}
\qquad \text{(A.27)}
$$

## SECOND LAW OF THERMODYNAMICS

The Clausius-Duheim inequality (law of entropy with viscous effects) can be obtained Gibbs' formula as

$$
\rho\frac{Ds}{Dt}\geq\frac{\rho}{\theta}\frac{De}{Dt}+\frac{p\rho}{\theta}\frac{D\rho^{-1}}{Dt} \qquad \text{(A.28)}
$$

Substituting from the energy conservation, A.31, the relation becomes

$$
\begin{aligned}
&\rho\frac{Ds}{Dt}\geq\frac{1}{\theta}\underline{\underline{\mathbf{t}}}:\nabla\mathbf{v}-\frac{p}{\rho\theta}\frac{D\rho}{Dt}-\frac{\nabla\cdot\dot{\mathbf{q}}}{\theta}+\frac{\rho h}{\theta}\\
&+\frac{\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}-\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}+\mathcal{J}\cdot\mathcal{E}}{\theta}
\end{aligned}
$$

(A.29)

By definition,

$$
\begin{aligned}
&\nabla\cdot\mathbf{v}=\mathrm{tr}\,\underline{\underline{\mathbf{d}}}\\
&\Phi=\underline{\underline{\tau}}:\underline{\underline{\mathbf{d}}}
\end{aligned}
\qquad \text{(A.30)}
$$

where tr is the trace of the matrix then the inequality becomes

$$
\begin{aligned}
&\rho\frac{Ds}{Dt}\geq\frac{\rho h+\Phi}{\theta}-\nabla\cdot\left(\frac{\dot{\mathbf{q}}}{\theta}\right)+\dot{\mathbf{q}}\cdot\nabla\left(\frac{1}{\theta}\right)\\
&+\frac{\rho\mathcal{E}\cdot\frac{D\left(\frac{\mathbf{P}}{\rho}\right)}{Dt}-\mathcal{M}\cdot\frac{D\mathbf{B}}{Dt}+\mathcal{J}\cdot\mathcal{E}}{\theta}
\end{aligned}
\qquad \text{(A.31)}
$$

Then the final expression for the Clausius-Duheim inequality is

$$\rho\frac{Ds}{Dt} \geq \frac{\rho h + \Phi}{\theta} - \nabla \cdot \left(\frac{\dot{\mathbf{q}}}{\theta}\right) - \frac{\dot{\mathbf{q}} \cdot \nabla\theta}{\theta^2} + \frac{\rho\mathcal{E} \cdot \dfrac{D\left(\dfrac{\mathbf{P}}{\rho}\right)}{Dt} - \mathcal{M} \cdot \dfrac{D\mathbf{B}}{Dt} + \mathcal{J} \cdot \mathcal{E}}{\theta} \tag{A.32}$$

Note that this is the same relationship which would be obtained from the global conservation law for the second law of thermodynamics and including viscous effects.