

**FED-Vol. 235/MD-Vol. 71, Developments in Electrorheological Flows  
ASME 1995**

**ELECTRO-MAGNETO-HYDRODYNAMICS:  
(PART 1) INTRODUCTORY CONCEPTS**

**George S. Dulikravich**

Department of Aerospace Engineering  
The Pennsylvania State University  
State College, Pennsylvania

**Sean R. Lynn**

Department of Aerospace Engineering  
The Pennsylvania State University  
State College, Pennsylvania

## ABSTRACT

Recent advances in numerical techniques and computing technology, as well as new, fully rigorous, theoretical treatments, have made analysis of combined electro-magneto-hydrodynamic (EMHD) flows well within reach. A survey of electro-magnetics and the theory describing combined electro-magneto-hydrodynamic flows is presented. In Part 1 emphasis is placed on describing the sources and interactions of electric and magnetic fields described by Maxwell's equations. The natural and induced sources of material polarization and magnetization are described as well as the sources and transport of electric charge in the fluid. The paper concludes with a presentation of the unified EMHD system of equations, combining Maxwell's equation, the Navier-Stokes equations and material constitutive relations. Part 2 of this paper will concentrate on the equations of fluid motion and the additional forces and energy imparted to them by electro-magnetic fields.

## NOMENCLATURE

<b>B</b>	= magnetic field vector, $\text{kg A}^{-1} \text{s}^{-2}$
<b><u>d</u></b>	= rate of deformation tensor, $\text{s}^{-1}$
<b>D</b>	= electric displacement field vector, $\text{A s m}^{-2}$
<b>e</b>	= internal energy per unit mass, $\text{m}^2 \text{s}^{-2}$
<b>E</b>	= electric field vector, $\text{kg m s}^{-3} \text{A}^{-1}$
<b><math>\mathcal{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}</math></b>	= electromotive intensity vector, $\text{kg m s}^{-3} \text{A}^{-1}$
<b>f</b>	= mechanical body force vector per unit mass, $\text{m s}^{-2}$
<b><math>\mathbf{F}^{\text{EM}}</math></b>	= electromagnetic body force vector per unit volume, $\text{kg m}^{-2} \text{s}^{-2}$
<b>h</b>	= heat source or sink per unit mass, $\text{m}^2 \text{s}^{-3}$
<b>H</b>	= magnetic field strength vector, $\text{A m}^{-1}$
<b><math>\mathbf{J}_0</math></b>	= apparent magnetization current vector, $\text{A m}^{-2}$
<b><math>\mathbf{J}_d</math></b>	= electric drift current vector, $\text{A m}^{-2}$
<b><math>\mathbf{J}_p</math></b>	= polarization current vector, $\text{A m}^{-2}$

<b><math>\mathbf{J} = \mathbf{J}_d + \mathcal{I}</math></b>	= electric current vector, $\text{A m}^{-2}$
<b><math>\mathcal{I}</math></b>	= electric conduction current vector, $\text{A m}^{-2}$
<b><math>\kappa</math></b>	= thermal conductivity coefficient, $\text{kg m s}^{-3} \text{K}^{-1}$
<b><math>\mathbf{M}_0</math></b>	= magnetization vector per unit volume due to rotational motion of charged particles, $\text{A m}^{-1}$
<b><math>\mathbf{M}_p</math></b>	= intrinsic or natural magnetization vector per unit volume, $\text{A m}^{-1}$
<b><math>\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_p</math></b>	= total magnetization vector per unit volume, $\text{A m}^{-1}$
<b><math>\mathcal{M} = \mathbf{M} + \mathbf{v} \times \mathbf{P}</math></b>	= magnetomotive intensity vector per unit volume, $\text{A m}^{-1}$
<b>p</b>	= pressure, $\text{kg m}^{-1} \text{s}^{-2}$
<b><math>\mathbf{P}_e</math></b>	= polarization vector per unit volume due to electric charge, $\text{A s m}^{-2}$
<b><math>\mathbf{P}_p</math></b>	= polarization vector per unit volume due to total dipole moments, $\text{A s m}^{-2}$
<b><math>\mathbf{P} = \mathbf{P}_e + \mathbf{P}_p</math></b>	= total polarization vector per unit volume, $\text{A s m}^{-2}$
<b><math>q_e</math></b>	= local free electric charge per unit volume, $\text{A s m}^{-3}$
<b><math>q'_e = -\nabla \cdot \mathbf{P}_e</math></b>	= inhomogeneous electric charge per unit volume, $\text{A s m}^{-3}$
<b><math>q_p = -\nabla \cdot \mathbf{P}_p</math></b>	= apparent electric charge per unit volume, $\text{A s m}^{-3}$
<b><math>\dot{q}</math></b>	= heat flux vector, $\text{kg s}^{-3}$
<b><math>q_0 = q_e - q'_e</math></b>	= free electric charge per unit volume, $\text{A s m}^{-3}$
<b>Q</b>	= point electric charge, $\text{A s}$
<b>s</b>	= entropy per unit mass, $\text{m}^2 \text{kg}^{-1} \text{K}^{-1} \text{s}^{-2}$
<b><math>\mathbf{t} = -\mathbf{p}\mathbf{I} + \mathbf{\tau}</math></b>	= hydrodynamic stress tensor, $\text{kg m}^{-1} \text{s}^{-2}$
<b>v</b>	= fluid velocity vector, $\text{m s}^{-1}$
<b><math>\mathcal{V}</math></b>	= volume, $\text{m}^3$

## Greek Symbols

$\epsilon$	= dielectric constant or electric permittivity coefficient, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_0$	= dielectric constant or electric permittivity of vacuum ( $\epsilon_0 = 8.854 \times 10^{-12}$ ), $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\sigma$	= electric conductivity coefficient, $\text{kg m s}^{-3} \text{A}^{-2}$
$\eta$	= shear viscosity coefficient, $\text{kg m}^{-1} \text{s}^{-1}$
$\theta$	= absolute temperature, K
$\rho$	= fluid density, $\text{kg m}^{-3}$
$\tau$	= viscous stress tensor, $\text{kg m}^{-1} \text{s}^{-2}$
$\mu$	= magnetic permeability coefficient, $\text{kg m A}^{-2} \text{s}^{-2}$
$\mu_0$	= magnetic permeability of vacuum, ( $\mu_0 = 4\pi \times 10^{-7}$ ), $\text{kg m A}^{-2} \text{s}^{-2}$
$\chi^E$	= relative electric permittivity or susceptibility ( $\chi^E = \epsilon/\epsilon_0 - 1$ ), nondimensional
$\chi^M$	= relative magnetic permeability or susceptibility ( $\chi^M = 1 - \mu_0/\mu$ ), nondimensional
$\Phi$	= viscous dissipation function, $\text{kg m}^{-1} \text{s}^{-3}$
$\Psi$	= material free energy function, $\text{m}^2 \text{s}^{-2}$

## 1. INTRODUCTION

The ability of electro-magnetic fields to influence fluid flow and heat transfer has long been known and used with varying degrees of success. The equations governing these flows consist of the Navier-Stokes equations of fluid motion coupled with Maxwell's equations of electro-magnetics and material constitutive relations. The field studying these flows is Electro-Magneto-Hydrodynamics (EMHD). The full system of equations has, up until recently, been far too complex to solve generally. The full Navier-Stokes relations are complex and become even more so when analysis of flows of realistic interest are desired (that is, turbulent, chemically reacting, multi-constituent, non-Newtonian and/or visco-elastic flows). Coupled with Maxwell's equations the complexity of the system is raised by orders of magnitude. Recently, rigorous theoretical continuum mechanics treatments of EHD (Wineman and Rajagopal, 1995) and unified EMHD flows (Eringen and Maugin, 1990) have allowed greater varieties of EMHD flows to be studied. These continuum mechanics approaches are limited to non-relativistic, quasi-static or relatively low frequency phenomenon (Bergman, 1962; Marcinkowski, 1992; Lakhtakia, 1993).

The objectives of this paper are to provide background resource and theory to allow implementation of numerical analysis for unified EMHD flows. The field is too vast to exhaustively cover the subject, so the intent is to provide an introductory survey of the field. To accomplish this, Part 1 provides an overview of electro-magnetic theory with concentrated effort placed on descriptions of the electric and magnetic fields and electric charge and current. Effort is also made to provide a physical understanding of the field-material interactions causing polarization and magnetization. Finally, the system of equations governing the unified EMHD theory is presented (Eringen and Maugin, 1990). Part 2 will discuss the equations of motion in more detail and will compare the unified EMHD theory with classical EHD and MHD models.

## 2. ELECTRIC FIELD

The concept of an electric field is typically developed from the *Coulomb force* between two electric point charges in free space (vacuum). The force,  $\mathbf{F}$ , between two point charges  $Q_1$  and  $Q_2$  is defined as (Cottingham and Greenwood, 1991, p. 15)

$$\mathbf{F} = \frac{Q_1 Q_2 (\bar{\mathbf{r}}_2 - \bar{\mathbf{r}}_1)}{4\pi\epsilon_0 |\bar{\mathbf{r}}_2 - \bar{\mathbf{r}}_1|^3} \quad (1)$$

where  $\bar{\mathbf{r}}_1$  and  $\bar{\mathbf{r}}_2$  are the position vectors of the stationary point charges  $Q_1$  and  $Q_2$  and  $\epsilon_0$  is the *permittivity of free space*. From this definition of electric force, the static *electric field*,  $\mathbf{E}$ , at  $\bar{\mathbf{r}}$  due to a single, stationary charge,  $Q_1$ , at position  $\bar{\mathbf{r}}_1$  is defined as (Johnk, 1988, p. 32)

$$\mathbf{E} = \frac{Q_1 (\bar{\mathbf{r}} - \bar{\mathbf{r}}_1)}{4\pi\epsilon_0 |\bar{\mathbf{r}} - \bar{\mathbf{r}}_1|^3} \quad (2)$$

This concept can be expanded to define the electric field due to multiple charges,  $Q_i$ , through the superposition principle (Cottingham and Greenwood, 1991, p.16). In this case the static electric field at the location  $\bar{\mathbf{r}}$  is defined as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i (\bar{\mathbf{r}} - \bar{\mathbf{r}}_i)}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_i|^3} \quad (3)$$

Thus, the electric field is a measure of the amount of force that can be exerted on a point charge by the electric charges surrounding it. Through concept of the electric field, no longer must the effect of each charge be considered individually.

If the electric field vector is integrated over the surface of a control volume,  $S$ , the result will be

$$\int_S \mathbf{E} \cdot \bar{\mathbf{n}} \, dS = - \int_V \nabla \cdot \left( \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_i|} \right) \cdot \bar{\mathbf{n}} \, dS \quad (4)$$

since (Cottingham and Greenwood, 1991, pp. 16-20)

$$\frac{(\bar{\mathbf{r}} - \bar{\mathbf{r}}_i)}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_i|^3} = -\nabla \left( \frac{1}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}_i|} \right) \quad (5)$$

After transforming the surface integral to a volume integral and rearranging, equation 4 becomes

$$\int_V \nabla \cdot (\epsilon_0 \mathbf{E}) \, d\mathcal{V} = \int_V \sum_i \frac{Q_i}{4\pi |\bar{\mathbf{r}} - \bar{\mathbf{r}}_i|^3} \, d\mathcal{V} \quad (6)$$

The result is the well known Gauss's law of electrostatics in a vacuum

$$\nabla \cdot (\epsilon_0 \mathbf{E}) = q_e \quad (7)$$

where  $q_e$ , the local free charge density, is defined as

$$q_e = \frac{\sum_i Q_i}{\mathcal{V}} \quad (8)$$

Gauss's law states that the electric field,  $\mathbf{E}$ , diverging from a point is related to the free charge density,  $q_e$ , at that point.

### 3. POLARIZATION

Previously, the interactions of charges and electric fields in a vacuum were considered. The presence of material media affects these interactions. In order to define these effects it is necessary to introduce the concept of polarization. Charge polarization or *polarization* is created when charges of opposite signs are separated by a distance. Thus, polarization is related to the orientation of charges in space and may occur in many forms. Atoms under the influence of an electric field may have a displacement of the negatively charged electron cloud and the positively charged nucleus which result in polarization. A molecule may have natural polarity due to its atomic geometry. An example of molecular polarization is a water molecule. Also, a solid or liquid continuum may have polarization due to an internal charge gradient. Similarly, electric fields passing through two different, contacting materials may cause polarization.

The existence of polarization creates a moment called the electric or dipole moment. The *dipole moment* is defined as the distance from the net negative to the net positive charge multiplied by the net positive charge. For a simple dipole with a single positive charge,  $Q^+$ , separated from a single negative charge,  $Q^-$ , by a distance,  $\xi$ , the dipole moment is defined as (Eringen and Maugin, 1990, p. 29)

$$\mathbf{p} = Q^+ \xi \quad (9)$$

An atomic dipole is created when the electron cloud of an atom is displaced from its nucleus through the action of an applied electric field (Figure 1).

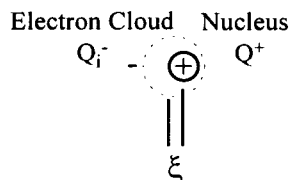


Figure 1. Example of an atomic dipole

A similar moment is produced in any atom, molecule, group of particles or continuum in which a net positive charge is separated from a net negative charge. In the case of a group of particles,

the encountered moment is the sum of the individual particle dipole moments. Of more importance to this discussion, however, is a discussion of polarization in a continuum. If  $\mathbf{p}$  is the dipole moment per unit volume and  $\mathcal{V}$  is the continuum volume, then the *continuum polarization* is defined as (Sutton and Sherman, 1965, p. 21)

$$\mathbf{P} = \frac{\int \mathbf{p} d\mathcal{V}}{\left(\int d\mathcal{V}\right)^2} \quad (10)$$

This definition holds regardless of whether the dipole moments arise from free electrons, ions, atoms, molecules or simply a charge gradient within a continuum.

Although many references define several sources of polarization (Pohl, 1978), there are essentially two main sources of polarization: induced and natural (Sutton and Sherman, 1965). *Induced polarization*,  $\mathbf{P}_p$ , is caused by an electric field acting on natural dipoles or neutral particles. The applied electric field induces an initial charge separation in neutral particles (Figure 2, adapted from Johnk, 1988). The applied electric field also creates greater charge separation within the molecules and causes molecular alignment with the applied field in natural dipoles (Figure 3, adapted from Johnk, 1988). The polarization shown in the first figure is wholly dependent on the electric field, whereas the second figure shows polarization which is dependent on both the electric field and on molecular geometry.

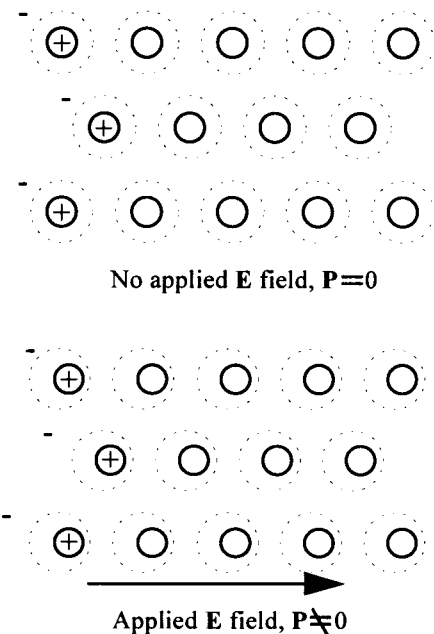


Figure 2. Induced polarization on initially non-polar molecules

In contrast, *natural polarization*,  $\mathbf{P}_e$ , arises from natural dipoles and charged particles. An example of a natural dipole is water which has a geometry such that one end of the molecule is more positive than the other. Figure 3 shows natural dipoles.

Ions, on the other hand are atoms or molecules whose overall charge is uneven; either positive or negative. At this point it is vital to realize that Figure 3 illustrates both types of polarization: both the natural polarization caused by the molecular geometry and induced polarization from the alignment of the molecules with the electric field. This requires further discussion.

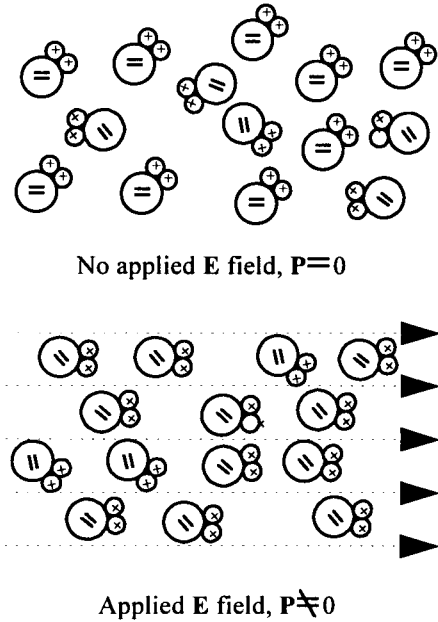


Figure 3. Induced polarization on initially polar molecules

Although the molecules in the upper half of Figure 3 have polarization on a particle or molecular level, they do not have polarization on a continuum level. Consider water,  $H_2O$ , as a liquid or vapor for instance (a geometry similar to that in Figure 3). On a molecular level water has polarization by virtue of its geometry. However, due to the fact that it is in a fluid state and the molecules are allowed to move freely and orient randomly, it will not have polarization on a continuum level. The sum of the molecular dipole moments throughout the continuum is zero. For water to be polarized on a continuum level an electric field must be applied as shown in the lower half of Figure 3.

Now consider the fluid water as it is frozen with an applied electric field. The molecules are no longer able to change their position or orientation. Even after the electric field is removed, the ice still has polarization on a continuum level. The polarization caused by the applied electric field aligning the water molecules is literally frozen into the ice. From this example it can be seen that even though a substance is polarized on a molecular level it need not be polarized at the level of a continuum.

From the above discussion it may seem that there is no reason, when dealing with fluids, to consider natural polarization. This, however, would be an erroneous assumption. Though the natural polarization may show no continuum effects without the presence of an electric field, in an electric field the overall polarization seen is both the induced polarization,  $P_p$ , due to the electric field and the natural polarization,  $P_e$ , of the molecules

(which are now aligned by the electric field). The *total polarization*,  $P$ , is defined as (Sutton and Sherman, 1965, p. 22)

$$P = P_p + P_e \quad (11)$$

Both types of polarization can produce a charge density within the continuum. This may be seen by taking an elemental control volume around a polarized continuum and performing a volumetric charge balance as performed by Johnk (1988, p. 118).

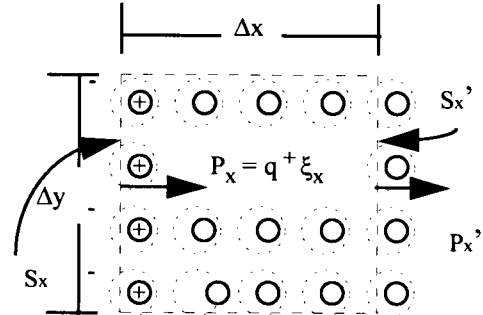


Figure 4. Volumetric charge balance for derivation of polarization charge

By summing the polarization into the control volume one obtains

$$\sum_{in} P = P_x \Delta y - P_x' \Delta y = -\frac{\partial q + \xi}{\partial x} \Delta x \Delta y = -\frac{\partial P_x}{\partial x} \Delta x \Delta y \quad (12)$$

This states that there is a net negative polarization or, in effect, a net negative charge within the control volume. This charge is designated  $q_{polarization}$ , the *polarization charge density*. When the preceding analysis is expanded to three dimensions the following relationship is obtained

$$q_{polarization} = -\nabla \cdot P \quad (13)$$

This derivation is valid for any type of polarization. Applying this definition to induced polarization,  $P_p$ , gives rise to the *apparent charge density*,  $q_p$ , which is defined as (Sutton and Sherman, 1965, p. 21)

$$q_p = -\nabla \cdot P_p \quad (14)$$

Similarly, natural polarization,  $P_e$ , gives rise to an *inhomogeneous charge density*,  $q_e'$ , which is defined as (Sutton and Sherman, 1965, p. 22)

$$q_e' = -\nabla \cdot P_e \quad (15)$$

The inhomogeneous charge density is caused by a non-uniform distribution of charges within the continuum.

From the dipole moment (equation 9) it can be seen that a time-varying distance,  $\xi$ , will produce a charge movement; in effect, an electric current. This *polarization current*,  $J_p$ , is

defined as the variation of the total polarization with respect to time (Johnk, 1988, p. 121)

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \quad (16)$$

From the above discussion it can be deduced that the degree to which material is polarized is related not only to the strength of the applied electric and magnetic fields, but to physical properties of the material expressed in terms of the material free energy,  $\Psi$ . The general constitutive relation for electro-static or low frequency polarization is defined as (Eringen and Maugin, 1990, p. 173)

$$\mathbf{P} = -2\rho \left( \frac{\partial \Psi}{\partial I_1} \boldsymbol{\varepsilon} + \frac{\partial \Psi}{\partial I_3} (\boldsymbol{\varepsilon} \cdot \mathbf{B}) \mathbf{B} \right) \quad (17)$$

where,

$$\Psi = \Psi(I_1, I_2, I_3, \theta, \rho^{-1}) \quad (18)$$

$$\boldsymbol{\varepsilon} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad (19)$$

$$I_1 = \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \quad (20)$$

$$I_2 = \mathbf{B} \cdot \mathbf{B} \quad (21)$$

$$I_3 = (\boldsymbol{\varepsilon} \cdot \mathbf{B})^2 \quad (22)$$

These relations, especially the free energy relation (equation 18) can be quite complex. Fortunately a large number of materials are relatively linear, meaning that polarization is a function of one material property and the strength and direction of the applied electric field (Haus and Melcher, 1989, pp. 220-222). For linear materials the free energy of the material becomes (Eringen and Maugin, 1990, p. 178)

$$\Psi = \Psi_0 - \frac{1}{2\rho} \left( \varepsilon_0 \chi^E I_1 + \frac{\chi^M}{\mu_0} I_2 \right) \quad (23)$$

The material constant,  $\chi^E$ , is the *dielectric susceptibility* and is typically obtained experimentally (Cottingham and Greenwood, 1991, p. 86). Thus, for linear materials only the first term in equation 17 remains and is usually written as (Eringen and Maugin, 1990, p. 178; Johnk, 1988, p. 164)

$$\mathbf{P} = -2\rho \frac{\partial \Psi}{\partial I_1} \boldsymbol{\varepsilon} = \varepsilon_0 \chi^E \boldsymbol{\varepsilon} \quad (24)$$

The dielectric susceptibility is often a constant value, although the medium in use and external factors, especially electro-magnetic frequency and temperature, may have vary its value (Haus and Melcher, 1989, pp. 220-222).

#### 4. MAGNETIC FIELD

The magnetic field is the second type of field of force originating from an electric charge. However, a derivation of the magnetic field is not as intuitively straightforward as the derivation for the electric field. There are several analogies that may be drawn between the two. First, both the electric field and magnetic field create forces.

The electric field is obtained from the Coulomb force between electric charges while the magnetic field,  $\mathbf{B}$ , is obtained from the ponderomotive force. The Coulomb and ponderomotive forces combine to form the Lorentz force, which is defined as (Johnk, 1988, p. 28)

$$\mathbf{F}^{EM} = q_0 \mathbf{E} + q_0 \mathbf{v} \times \mathbf{B} \quad (25)$$

where the first term is the Coulomb force. The second term states that the velocity of the free charge  $q_0$ , the magnetic field,  $\mathbf{B}$ , and the resulting force are all mutually perpendicular.

Analogous to the electric field relation to electric charge, the magnetic field is related to charge movement or current,  $\mathbf{J}$ . The fundamental statement to this effect is called Ampere's law and, for the case of steady magnetic field in a vacuum without polarization or magnetization, is expressed as (Johnk, 1988, p. 85)

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} \quad (26)$$

This law states that a steady current,  $\mathbf{J}$ , will produce a circulating or rotational magnetic field,  $\mathbf{B}$ .

#### 5. MAGNETIZATION

Magnetization is the magnetic field analogy of the electric field's polarization. Magnetization comes from two sources: particle circular motion and intrinsic or natural particle magnetism. To derive the magnetization due to circular motion of the particle, its velocity vector,  $\mathbf{v}$ , may be expressed as a sum of the linear drift velocity,  $\mathbf{v}_d$ , and a rotational velocity,  $\omega \times \mathbf{r}$  (Sutton and Sherman, 1965, p. 25).

$$\mathbf{v} = \mathbf{v}_d + \omega \times \mathbf{r} \quad (27)$$

For a particle, the magnetization moment,  $\mathbf{v}$ , due to this rotational velocity is defined as (Sutton and Sherman, 1965, p. 26)

$$\mathbf{v} = \frac{Qr^2\omega}{2} \quad (28)$$

where  $Q$  is the particle charge,  $r$  is the distance from the particle to its center of rotation, and  $\omega$  is the angular velocity vector. Similar to the way continuum polarization was defined, continuum

magnetization due to circular motion is the average value of the particle magnetization (Sutton and Sherman, 1965, p. 26).

$$\mathbf{M}_v = \frac{\int v d\mathcal{V}}{\left(\int d\mathcal{V}\right)^2} \quad (29)$$

*Intrinsic* or *natural magnetization*,  $\mathbf{M}_p$ , is not associated with circular motion of the particle itself. Instead, intrinsic magnetization is associated with electron and nuclear spins; motion of the particle around a center to which it is attached by atomic forces. Thus, intrinsic magnetization is a material property. The *total magnetization*, then, is the sum of the magnetization from particle circular motion and that from intrinsic spin of the particle and may be written as

$$\mathbf{M} = \mathbf{M}_v + \mathbf{M}_p \quad (30)$$

Neither type of magnetization gives rise to charge densities as polarization did, but magnetization due to circular motion of the particles does create an electric current, the *apparent magnetization current*, which is defined as (Sutton and Sherman, 1965, p. 26)

$$\mathbf{J}_v = \nabla \times \mathbf{M}_v \quad (31)$$

Intrinsic or natural magnetization does not have a related current because electric current is generally defined as the macroscopic or external movement of charges. Because intrinsic or natural magnetization is caused by internal movement of charges, that is electrons attached to the nucleus, there is no electric current created (Sutton and Sherman, 1965, p. 26). At this point it is important to note that the charged particle drift velocity,  $\mathbf{v}_d$ , in equation 27 creates the convection or drift current,  $\mathbf{J}_d$ . The *drift current* is defined as (Sutton and Sherman, 1965, p. 67)

$$\mathbf{J}_d = q_0 \mathbf{v}_d \quad (32)$$

The drift current is not associated with magnetization, but with the physical linear motion of charged particles.

Magnetization, like polarization, depends not only on the electromagnetic field strengths, but on material properties as well. The constitutive definition of magnetization for steady or relatively low frequencies is defined as (Eringen and Maugin, 1990, pp. 177, 130)

$$\mathcal{M} = -2\rho \left( \frac{\partial \Psi}{\partial I_2} \mathbf{B} + \frac{\partial \Psi}{\partial I_3} (\boldsymbol{\varepsilon} \cdot \mathbf{B}) \mathbf{B} \right) \quad (33)$$

where

$$\mathcal{M} = \mathbf{M} + \mathbf{v} \times \mathbf{P} \quad (34)$$

The other variables have been previously defined in equations 18-22. If the material in question may be considered linear, that is the magnetization is a function of one material property and the strength and direction of the applied magnetic field, then the

magnetization is defined as (Eringen and Maugin, 1990, p. 178; Johnk, 1988, p. 164)

$$\mathcal{M} = -2\rho \frac{\partial \Psi}{\partial I_2} \mathbf{B} = \frac{\chi^M}{\mu_0} \mathbf{B} \quad (35)$$

where  $\chi^M$  is the *magnetic susceptibility*, and is usually obtained experimentally. This expression is similar to equation 24 for polarization of linear materials. Haus and Melcher (1989, pp. 371-377) and Cottingham and Greenwood (1991, pp. 92-96) discuss material aspects of magnetization in more detail. As with dielectric susceptibility, magnetic susceptibility may be a function of electro-magnetic frequency, temperature and other physical conditions.

## 6. ELECTRIC CHARGES

The purpose of this section is to categorize and define the sources of electric charges and to combine these charges into the free charge density,  $q_0$ , found in Gauss's Law for a polarizable material medium. Electric charges come in two types: free and bound. Free charges arise from electrons in the outer or free atomic shells and from ions. Bound charges are those arising from the molecular geometry and displacement of atomic inner electron shells. Polar molecules are one example of bound charges.

Gauss's law without polarization and in a vacuum states that

$$\nabla \cdot (\boldsymbol{\varepsilon}_0 \mathbf{E}) = q_e \quad (36)$$

where  $q_e$  is the *local free charge density* caused by free ions and electrons. Polarization occurs as the applied electric field passes through a medium. The apparent and inhomogeneous charge densities, noted in equations 14 and 15, are caused by polarization and must be added to Gauss's law. The local free charge density,  $q_e$ , accounts for all the free electrons and ions in the continuum.

Knowing that the inhomogeneous charge density,  $q_e'$ , is caused by a free charge gradient within the continuum, it can be seen that the inhomogeneous charge density is part of the local charge density. By using the definition of inhomogeneous charge density due to natural polarization from equation 15, the inhomogeneous polarization charge density can be explicitly shown in Gauss's law as

$$\nabla \cdot (\boldsymbol{\varepsilon}_0 \mathbf{E}) = q_e - \left( q_e' + \nabla \cdot \mathbf{P}_e \right) \quad (37)$$

Note that the second and third terms add to zero. The only other source of charge is the apparent charge from induced polarization,  $q_p$ , given in equation 14. This may be added directly to the free charge density to form

$$\nabla \cdot (\boldsymbol{\varepsilon}_0 \mathbf{E}) = q_e - q_e' - \nabla \cdot \mathbf{P}_e - \nabla \cdot \mathbf{P}_p \quad (38)$$

Noting from equation 11 that the total polarization,  $\mathbf{P}$ , is the sum of the inhomogeneous and natural polarizations, the polarization charge terms may be grouped together to form

$$\nabla \cdot (\epsilon_0 \mathbf{E}) = q_e - q_e' - \nabla \cdot \mathbf{P} \quad (39)$$

The local free charge density minus the inhomogeneous charge density is, by definition, the *total or free charge density*,  $q_o$ , defined as (Sutton and Sherman, 1965, p. 21)

$$q_o = q_e - q_e' \quad (40)$$

Introducing the free charge density and grouping the polarization charge density with the electric field, Gauss's Law becomes

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = q_o \quad (41)$$

Usually the electric field vector and the polarization vector are combined to form the *electric displacement* or *electrical induction vector*,  $\mathbf{D}$ , which is defined as (Johnk, 1988, p. 164)

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (42)$$

Gauss's Law then becomes (Sutton and Sherman, 1965, p. 22)

$$\nabla \cdot \mathbf{D} = q_o \quad (43)$$

Note that if polarization is a linear function of the steady or relatively low frequency electric field,  $\mathbf{E}$ , then the displacement vector becomes

$$\mathbf{D} = \epsilon \mathbf{E} \quad (44)$$

where  $\epsilon$  is the *dielectric permittivity* and is derived by substituting equation 24 into equation 42 and is shown as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi^E) \mathbf{E} = \epsilon \mathbf{E} \quad (45)$$

## 7. ELECTRIC CURRENT

Analogous to the definition of free charge density through the assembly of Gauss's law, the total current will be defined through the assembly of the Ampere-Maxwell law. In addition to the currents arising from magnetization and direct charge motion defined in equations 31 and 32, other conduction currents,  $\mathcal{J}$ , have been observed and must be taken into account when defining the total current,  $\mathbf{J}$  (Eringen and Maugin, 1990, pp. 161-163). These currents are caused by the Seebeck, Hall, Nernst and a host of other effects which will be described in greater detail in Part 2 of this paper. With this in mind, the Ampere-Maxwell law with no magnetization or polarization states that (Johnk, 1988, p. 86)

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} + \mathbf{J}_d + \mathcal{J} \quad (46)$$

where the magnetic field,  $\mathbf{B}$ , the drift current  $\mathbf{J}_d$ , and the electric field,  $\mathbf{E}$ , have previously been defined. Note that this is different than the Ampere's law in equation 26. This is due to the fact that equation 26 was for magnetostatics; the case where the applied electric and magnetic fields are constant. Introducing the effects of magnetization and polarization and rearranging constants, the Ampere-Maxwell law of electrodynamics may be rewritten as (Sutton and Sherman, 1965, p. 30)

$$\nabla \times \mathbf{B} = \mu_0 \left( \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} + \mathbf{J}_d + \mathcal{J} \right) \quad (47)$$

From equation 30 it can be seen that the total magnetization,  $\mathbf{M}$ , is the sum of the circular motion and intrinsic or natural magnetizations. The intrinsic magnetization does not give rise to a current since  $\nabla \times \mathbf{M}_p = 0$ . Thus the magnetization current could be written in terms of the circular motion magnetization,  $\mathbf{M}_c$ , only. The polarization current is defined in equation 16 as the time derivative of the polarization vector. Thus, the third term on the right hand side of equation 47 and the polarization current may be combined into the electrical displacement vector,  $\mathbf{D}$ . Additionally, the magnetization and magnetic field vectors are often combined to form the magnetic field strength vector,  $\mathbf{H}$ , shown as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (48)$$

Implementing these two changes, the Ampere-Maxwell law for electrodynamics of polarizable and magnetizable media can be written as (Johnk, 1988, p. 147)

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_d + \mathcal{J} \quad (49)$$

At times it is necessary to define the total current,  $\mathbf{J}$ , as the sum of the apparent magnetization, drift and phenomenological currents (Sutton and Sherman, 1965, p. 26)

$$\mathbf{J} = \mathbf{J}_d + \mathcal{J} \quad (50)$$

since the contribution to the magnetization current by intrinsic magnetization is zero. The Ampere-Maxwell law for polarizable, magnetizable media can be written as (Johnk, 1988, p. 132)

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \mathbf{J}_v + \frac{\partial \mathbf{D}}{\partial t} \quad (51)$$

## 8. GOVERNING SYSTEM OF EQUATIONS

The full system of equations governing unified EMHD flow consists of the Maxwell's equations governing electromagnetism, the Navier-Stokes equations governing fluid flow and constitutive equations describing material behavior. This set of 27 equations allows solving for the 27 unknowns: fluid density ( $\rho$ ), charge density ( $q_o$ ), temperature ( $\theta$ ), pressure ( $p$ ), and the three vector components of velocity field ( $\mathbf{v}$ ), electric



field ( $\mathbf{E}$ ), magnetic field ( $\mathbf{B}$ ), magnetization ( $\mathbf{M}$ ), polarization ( $\mathbf{P}$ ), current ( $\mathbf{J}$ ), heat flux ( $\mathbf{q}$ ), and conduction current ( $\mathcal{J}$ ). The system of equations is made up by the set of equations 17, 33, 52, 53, 55, 56, 57, 58, 59, 61, 62, 63, 64 and an equation of state. Note that the conservation of electric charge has been grouped with Maxwell's equations, although it is not traditionally part of the system.

### 8.1 Maxwell Equations

Maxwell's equations are the system of linear differential equations governing electro-magnetic fields. They are given as (Cottingham and Greenwood, 1991, p. xv)

#### Gauss' law

$$\nabla \cdot \mathbf{D} = q_0 \quad (52)$$

#### Ampere-Maxwell law

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad (53)$$

or

$$\nabla \times \mathbf{B} = \mu_0 \left( \nabla \times \mathbf{M} + \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \quad (54)$$

#### Conservation of magnetic flux

$$\nabla \cdot \mathbf{B} = 0 \quad (55)$$

#### Faraday's law

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (56)$$

#### Conservation of electric charges (from equations 52 and 53)

$$\frac{\partial q_0}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (57)$$

Detailed descriptions of these equations can be found in any number of texts (Johnk, 1988; Cottingham and Greenwood, 1991; Haus and Melcher, 1989). Only a brief description of some of the implications of these equations is given here.

Two of the physical implications of Gauss's law are that electric field lines radiate from a charge and are influenced by polarization. This was discussed in section 6 of this paper. The magnetic counterpart to Gauss's law is the conservation of magnetic flux. It may be interpreted as the fact that magnetic field lines form closed loops instead of ending in space. This law is a consequence of the fact that no magnetic monopoles have ever been found. Faraday's law may be physically interpreted as the fact that an electric field induces a time-varying magnetic field, or equivalently, a time-varying magnetic field produces a steady electric field. The last of the typical set of Maxwell's equations is the Ampere-Maxwell law which which is discussed in section 7 may be interpreted as the fact that a steady magnetic field produces a time-varying electric field (magnetic analogy of

Faraday's law). Additionally, a magnetic field induces a charge current. The last equation included in the system of equations governing electro-magnetic fields is the conservation equation of the local free charge which states that in the absence of chemical or atomic reactions, charges are neither created or destroyed. Thus, the only change in charges comes from their motion into and out of the point of reference.

### 8.2 NAVIER-STOKES EQUATIONS

The equations of motion governing EMHD flow are the Navier-Stokes relations into which electromagnetic effects have been included. A summary of these equations is given below with a derivation from the global conservation law governing continuum mechanics given in Part 2 of this paper. A rigorous derivation of these equations for electro-magnetic fluids is completed by Eringen and Maugin (1990).

#### Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (58)$$

#### Conservation of momentum

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \underline{\underline{t}}) - \rho \mathbf{f} - \mathbf{F}^{\text{EM}} = 0 \quad (59)$$

where the electromagnetic force per unit volume is

$$\begin{aligned} \mathbf{F}^{\text{EM}} = & q_0 \mathbf{E} + \mathbf{J} \times \mathbf{B} + (\nabla \mathbf{E}) \cdot \mathbf{P} + (\nabla \mathbf{B}) \cdot \mathbf{M} \\ & + \nabla \cdot ((\mathbf{P} \times \mathbf{B}) \mathbf{v}) + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}) \end{aligned} \quad (60)$$

#### Conservation of energy

$$\begin{aligned} \rho \frac{D\epsilon}{Dt} + \frac{1}{2} \frac{\partial (\rho v^2)}{\partial t} + \nabla \cdot \left( \mathbf{v} \frac{1}{2} \rho v^2 \right) - \rho \frac{Dv}{Dt} \cdot \mathbf{v} \\ - \nabla \cdot (\underline{\underline{t}} \cdot \mathbf{v}) + (\nabla \cdot \underline{\underline{t}}) \cdot \mathbf{v} + \nabla \cdot \mathbf{q} - \rho h \\ - \rho \mathcal{E} \cdot \frac{D(\frac{\mathbf{P}}{\rho})}{Dt} + \mathcal{M} \cdot \frac{D\mathbf{B}}{Dt} - \mathcal{J} \cdot \mathcal{E} = 0 \end{aligned} \quad (61)$$

#### Clausius-Duheim Inequality

$$\begin{aligned} \rho \frac{Ds}{Dt} \geq & \frac{\rho h + \Phi}{\theta} - \nabla \cdot \left( \frac{\mathbf{q}}{\theta} \right) - \frac{\mathbf{q} \cdot \nabla \theta}{\theta^2} \\ & + \frac{\rho \mathcal{E} \cdot \frac{D(\frac{\mathbf{P}}{\rho})}{Dt} - \mathcal{M} \cdot \frac{D\mathbf{B}}{Dt} + \mathcal{J} \cdot \mathcal{E}}{\theta} \end{aligned} \quad (62)$$

where

$$\begin{aligned}
\mathcal{J} = & (\sigma_1 \boldsymbol{\varepsilon} + \sigma_3 \underline{\underline{\mathbf{d}}} \cdot \boldsymbol{\varepsilon} + \sigma_7 \underline{\underline{\mathbf{d}}}^2 \cdot \boldsymbol{\varepsilon}) \\
& + (\sigma_2 \nabla \theta + \sigma_4 \underline{\underline{\mathbf{d}}} \cdot \nabla \theta + \sigma_8 \underline{\underline{\mathbf{d}}}^2 \cdot \nabla \theta) \\
& + \left[ \sigma_3 \boldsymbol{\varepsilon} \times \mathbf{B} + \sigma_{11} (\underline{\underline{\mathbf{d}}} \cdot (\boldsymbol{\varepsilon} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \cdot \boldsymbol{\varepsilon}) \times \mathbf{B}) \right] \\
& + \left[ \sigma_6 \nabla \theta \times \mathbf{B} + \sigma_{12} (\underline{\underline{\mathbf{d}}} \cdot (\nabla \theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \cdot \nabla \theta) \times \mathbf{B}) \right] \\
& + \sigma_9 (\mathbf{B} \cdot \boldsymbol{\varepsilon}) \mathbf{B} + \sigma_{10} (\mathbf{B} \cdot \nabla \theta) \mathbf{B}
\end{aligned} \quad (63)$$

$$\begin{aligned}
\dot{\mathbf{q}} = & (\kappa_2 \boldsymbol{\varepsilon} + \kappa_4 \underline{\underline{\mathbf{d}}} \cdot \boldsymbol{\varepsilon} + \kappa_8 \underline{\underline{\mathbf{d}}}^2 \cdot \boldsymbol{\varepsilon}) \\
& + (\kappa_1 \nabla \theta + \kappa_3 \underline{\underline{\mathbf{d}}} \cdot \nabla \theta + \kappa_7 \underline{\underline{\mathbf{d}}}^2 \cdot \nabla \theta) \\
& + \left[ \kappa_6 \boldsymbol{\varepsilon} \times \mathbf{B} + \kappa_{12} (\underline{\underline{\mathbf{d}}} \cdot (\boldsymbol{\varepsilon} \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \cdot \boldsymbol{\varepsilon}) \times \mathbf{B}) \right] \\
& + \left[ \kappa_5 \nabla \theta \times \mathbf{B} + \kappa_{11} (\underline{\underline{\mathbf{d}}} \cdot (\nabla \theta \times \mathbf{B}) - (\underline{\underline{\mathbf{d}}} \cdot \nabla \theta) \times \mathbf{B}) \right] \\
& + \kappa_9 (\mathbf{B} \cdot \nabla \theta) \mathbf{B} + \kappa_{10} (\mathbf{B} \cdot \boldsymbol{\varepsilon}) \mathbf{B}
\end{aligned} \quad (64)$$

Here  $\sigma_i$  and  $\kappa_i$  ( $i = 1-12$ ) are the physical properties of the media. In this case  $\sigma_1$  and  $\kappa_1$  are the electric and heat conductivities respectively. Note that  $\mathcal{J} = \mathbf{J} - \mathbf{J}_d$  and is the time-invariant form of the electric current and is called the conduction current (Eringen and Maugin, 1990, p. 53). In the above relations, the tensor  $\underline{\underline{\mathbf{d}}}$  is the rate of deformation tensor given as (Eringen and Maugin, 1990, p. 13)

$$\underline{\underline{\mathbf{d}}} = \frac{1}{2} (\mathbf{v}_{i,j} + \mathbf{v}_{j,i}) \quad (65)$$

Additionally,  $\underline{\underline{\mathbf{d}}}^2 = \underline{\underline{\mathbf{d}}} \cdot \underline{\underline{\mathbf{d}}}$  is the square of the rate of deformation tensor. Here it is important to note that the same relationship (equation 63) can be obtained by considering the second law of thermodynamics and including viscous effects.

## 9. CONCLUSION

The objective of this paper was to survey background theory to allow initial implementation of a unified EMHD theory presented. To accomplish this, Part I presented introductory concepts in electro-magnetic field theory. The sources of both the electric and magnetic fields were derived with emphasis placed on describing physical aspects of polarization and magnetization. Polarization was shown to arise from both natural and induced sources and magnetization was shown to come about from material intrinsic and particle circular motion. The non-linear relations of both polarization and magnetization were presented. However, many materials have linear material properties, considerably reducing the complexity of calculating polarization and magnetization. The sources of electric charge and electric current were discussed. Finally, the equations governing unified EMHD flows as derived by Eringen and

for current, heat transfer and material equation of state. Part 2 will discuss these equations of motion in more detail, concentrating on the material constitutive relations for electro-magnetic force, electric current and heat transfer. A detailed comparison of classical EHD and MHD models with the unified EMHD model will also be presented in Part 2.

## ACKNOWLEDGEMENTS

The authors would like to thank NASA Marshall Research Center Microgravity Program, particularly Dr. Martin Volz, for financial support of the second author. Thanks are also due to Professor A. Lakhtakia of Penn State for his valuable knowledge and suggestions.

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Maugin were presented. The system contains 27 equations, coupling Maxwell's equations of electro-magnetic fields, the Navier-Stokes flow-field equations and the constitutive relations

GEORGE S. DULIKRAVICH

FED-Vol. 235  
MD-Vol. 71

# DEVELOPMENTS IN ELECTRORHEOLOGICAL FLOWS – 1995 –

**presented at**

THE 1995 ASME INTERNATIONAL  
MECHANICAL ENGINEERING CONGRESS AND EXPOSITION  
NOVEMBER 12–17, 1995  
SAN FRANCISCO, CALIFORNIA

**sponsored by**

THE FLUIDS ENGINEERING DIVISION, ASME  
THE MATERIALS DIVISION, ASME

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THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS  
UNITED ENGINEERING CENTER / 345 EAST 47TH STREET / NEW YORK, N.Y. 10017