HTD-Vol. 312

Proceedings of the 30th

1995 NATIONAL HEAT TRANSFER CONFERENCE

VOLUME 10

- CONJUGATE HEAT TRANSFER, INVERSE PROBLEMS, AND OPTIMIZATION
- INVERSE PROBLEMS IN HEAT TRANSFER

presented at

THE 1995 NATIONAL HEAT TRANSFER CONFERENCE
PORTLAND, OREGON
AUGUST 6–8, 1995

sponsored by

THE HEAT TRANSFER DIVISION, ASME

edited by

WILLIAM J. BRYAN
ANSYS, INC.

JAMES V. BECK
MICHIGAN STATE UNIVERSITY

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
United Engineering Center • 345 East 47th Street • New York N.Y. 10017

HTD-Vol. 312, 1995 National Heat Transfer Conference – Volume 10
ASME 1995

INVERSE DETERMINATION OF BOUNDARY CONDITIONS IN STEADY HEAT CONDUCTION WITH HEAT GENERATION

Thomas J. Martin
Department of Aerospace Engineering
The Pennsylvania State University
University Park, PA

George S. Dulikravich
Department of Aerospace Engineering
The Pennsylvania State University
University Park, PA

ABSTRACT

Our unique inverse methodology for finding unknown boundary conditions for Laplace equation utilizing the Boundary Element Method (BEM) has been extended to the solution of two-dimensional inverse (ill-posed) Poisson problem of steady heat conduction with heat sources and sinks. The procedure is simple, reliable, non-iterative and cost effective. Accurate results in two-dimensional heat conduction with arbitrary distributions of heat sources have been obtained for several test cases where boundary conditions were unknown on certain boundaries. Because of its non-iterative, direct nature, our algorithm does not amplify errors in the over-specified input data supplied to parts of the boundary. Furthermore, it does not require regularization schemes, extrapolation to the boundary or mollification to suppress the amplification of input errors. Instead, a straight-forward modification to the BEM produces a single, highly singular solution matrix which we solved using a singular value decomposition matrix solver. Our method for the solution of ill-posed boundary condition problems governed by the Poisson equation also accepts input data at isolated interior points.

INTRODUCTION

The integrity of energy producing or consuming devices depends on maintaining an acceptable operating temperature by proper heat transfer. For example, Joule heating generated inside the electronic components strongly depends on the frequency of the alternating electromagnetic field and the local material properties. Another example of such domain-distributed heat sources is the microwave heating of food and materials processing. In the case of a buried nuclear or chemical toxic waste, the heat generated by ongoing reactions will be distributed throughout the burial site. In general, the internal heat generation may lead to local overheating, potentially serious equipment failures, and environmentally disastrous consequences. In order to understand the steady thermal field in these problems, a well-posed boundary value problem is often constructed, governed by the Poisson equation. It requires either temperature, heat flux or convective heat transfer coefficient specification over all boundaries of the domain. Well-posed Poisson problems also require the specification of heat source intensities throughout the domain.

However, local measurements and continuous monitoring of the heat sources in the entire domain are often impractical. This is because of the intrusive nature of a large number of sensors. Using sensors may be impossible to achieve in practice because of the highly volatile environment, as in the case of a waste dump, or because of the prohibitively small size of the domain available for placement of sensors as in the case of a computer chip. Thus, in many cases we are forced to solve an ill-posed boundary value problem where no data is available on certain boundaries. In other words, it would be highly desirable to develop a non-intrusive monitoring technique capable of utilizing over-determined thermal measurements on accessible boundaries of the domain. The over-specified boundary conditions would then be used to predict temperatures, heat fluxes, and convective heat transfer coefficients on the inaccessible boundaries. This objective is termed the steady inverse heat conduction problem (SIHCP) since it calls for the solution of the Laplace or Poisson equation for a steady temperature field subject to partially overspecified and partially underspecified (unavailable) thermal boundary conditions.

We have developed a non-iterative algorithm that can reliably and efficiently solve inverse boundary condition (ill-posed) problems governed by the Laplace equation on two-dimensional multiply-connected domains including regions with different temperature-dependent material properties (Martin and Dulikravich, 1993; 1994a; 1994b; Dulikravich and Martin, 1994a). An extended version of this method was also successfully used in solving ill-posed problems in two-dimensional elasticity (Martin, Halderman and Dulikravich, 1994). There, no boundary conditions for traction and deformation vectors were known on some of the boundaries while both traction and deformation vectors were specified on the remaining boundaries.

In this paper we elaborate on the extension of our method for the solution of the inverse (ill-posed) boundary condition problems to the Poisson equation involving an arbitrary known distribution of heat sources or sinks throughout the domain. This objective is different from the objective of a more common inverse shape (domain) determination problem (Dulikravich and Martin, 1994b) and the unsteady (Beck et al., 1984) inverse (ill-posed) heat conduction problem (UIHCP). The major concern when attempting to solve the UIHCP computationally has been with the automatic filtering of noisy data in the discrete thermocouple measurements. All measurement data errors, as well as numerical round-off errors, are amplified by the typical UIHCP algorithms. These numerical methods are usually formulated in the least squares sense where the overall error between the computed and measured temperatures is minimized (Kagawa et al., 1995). Among others, the method of regulizers, discrete mollification (Murio, 1993) against a suitable averaging kernel and other filtering techniques have also been implemented in order to smooth the extrapolated boundary values. To date, many of the UIHCP solutions were performed for specific geometries and cannot be readily extended to complex geometries. In fact, most attention has been focused on the one-dimensional UIHCP. Another basic concern is that relatively few UIHCP techniques used in engineering provide a quantitative method for determining what effect their smoothing operations have on the accuracy of the estimates (Hensel and Hills, 1986). Our new method solves a steady-state inverse (ill-posed) boundary condition problem governed by the Poisson equation and does not utilize any of the approaches used in UIHCP's.

The theory behind our SIHCP method is based on the Green's function solution method, commonly referred to as the Boundary Element Method (BEM). It is an integral technique that generates a set of linear algebraic equations with unknowns confined only to the boundaries. For well-posed problems, the resulting solution matrix can be solved by Gaussian elimination or any other standard matrix inverter. When an ill-posed problem is encountered, the matrix becomes highly singular. We have shown that the proper solution to this singular matrix provides accurate results to various SIHCPs. Our method has also been shown to be quite insensitive to measurement errors (Martin and Dulikravich, 1994a; 1994b) in the specified boundary conditions. The approach is somewhat similar, at least in theory, to those delivered by Backus and Gilbert (1970) and Lanczos (1961). These authors have discussed techniques which allow one to selectively discard eigenvalues and eigenvectors of a particular system of equations that tend to magnify errors.

NUMERICAL FORMULATION

This paper elaborates on our simple, robust and fast numerical solution algorithm to the ill-posed two-dimensional Poisson equation using the BEM. The algorithm is applicable to complex, multiply-connected, two and three-dimensional geometries as well as to inverse initial value problems. Temperature and heat flux data are not required on those boundaries where such measurements cannot be obtained. Instead, over-specified measurement data involving both temperature and heat flux are required on the parts of remaining, more accessible boundaries or at a small number of points within the domain.

The BEM (Brebbia, 1978) is based upon a Green's function solution procedure. It has been proven to be the most efficient numerical technique for solving linear boundary value problems such as those governing heat conduction, elasticity, wave propagation and electromagnetic fields. The BEM has recently been used to solve nonlinear problems such as viscous fluid flow, heat conduction with radiation, moving boundary problems such as solidification and melting, conjugate heat transfer, and other phenomenon. Our method of solving the SIHCP uses the BEM because this numerical technique has certain distinct advantages over the more common finite element or finite differencing methods. Analytic solutions to the partial differential equation, in the form of the Green's function, are part of the BEM solution. Therefore, high accuracy is expected with the BEM because introducing the Green's functions does not introduce any error into the solution. The BEM does not, like the finite element method, neglect the inter-element continuity terms. The degrees of freedom of the system are reduced such that unknowns are strictly confined to domain boundaries. The noniterative nature of the BEM eliminates reliability and convergence problems.

The governing equation for steady-state heat conduction in a solid with temperature-dependent coefficient of thermal conductivity, k(T), and arbitrarily distributed heat sources or sinks, g(x,y), per unit volume is

$$\nabla \cdot \big(k(T)\nabla T\big) + g(x,y) = 0 \qquad (1)$$

where T is the temperature. Equation (1) can be linearized by the application of the classical Kirchoff transformation which defines the heat function, u, as

$$u = \int_0^T \frac{k(T)}{k_0}\, dT \qquad (2)$$

where k_0 is the reference coefficient of thermal conductivity. Equation (1) is subsequently transformed into Poisson equation operating on the heat function, u. The boundary integral equation (BIE) for the Poisson equation is obtained from the weighted residual statement or Green's Theorem

$$c(x)u(x) + \int_\Gamma q^*(x,\xi)\, u(\xi)\, d\Gamma =$$

$$\int_\Gamma u^*(x,\xi)\, q(\xi)\, d\Gamma + \int_\Omega u^*(x,\xi)\, g(\xi)\, d\Omega \qquad (3)$$

The integration here is over all the boundaries, $\Gamma = \Gamma_1 + \Gamma_2 + ... + \Gamma_N$, of a multiply-connected domain. Here, $q = \partial u/\partial n$

is the heat flux, u^* is the fundamental Green's function solution (Brebbia and Dominguez, 1989) so that $q^* = \partial u^*/\partial n$, $\mathbf{n}$ is the direction of the outward normal to the boundary Γ, while $c(\mathbf{x})$ is a free term arising from the Dirac's delta function and the integration over the singularity in the sense of the Cauchy principal value at the point $\mathbf{x}$. Consequently, $c(\mathbf{x}) = 0.0$ when $\mathbf{x}$ is outside the domain, $c(\mathbf{x}) = 1.0$ when $\mathbf{x}$ is inside the domain, and $c(\mathbf{x}) = \theta / 2\pi$ when $\mathbf{x}$ is on the boundary. Here, θ is the internal angle of a corner between two neighboring boundary panels.

The fundamental solution for the two-dimensional Poisson equation is

$$u^* = \frac{1}{2\pi}\ln\left(\frac{1}{|x - \xi|}\right) \tag{4}$$

where ξ is the point of integration and $\mathbf{x}$ is the control point. The boundary can be discretized into N boundary panels connected at N boundary points. The functions u and q are interpolated linearly between their values at the end points of each boundary panel. This results in a set of N boundary integral equations, one for each of the N control points on the boundary. If temperature is known at discrete locations within the domain, additional equations can be added to this set, one for each additional control point. In addition, heat fluxes are allowed to be double-valued at corners formed by each pair of neighboring boundary panels. This formulation requires the domain, Ω, to be discretized into N_{VC} cells sharing both domain and boundary points in order to evaluate the field source integral. Since the unknowns are only on the boundary, the set of BIE's can be arranged in the following matrix form.

$$[\mathbf{H}]\,\{\mathbf{U}\} = [\mathbf{G}]\,\{\mathbf{Q}\} + \{\mathbf{P}\} \tag{5}$$

Without using internal temperature measurements, there will be a total of 2N unknowns in the equation set. For a well-posed boundary value problem, every point on the boundary is given one Dirichlet, Neumann or Robin-type boundary condition. The equation set that will result has N unknowns and N equations. The boundary conditions may be multiplied out on the right hand side and added to $\{\mathbf{P}\}$ to form a vector of knowns, $\{\mathbf{F}\}$, while the left-hand side remains in the standard form $[\mathbf{A}]\{\mathbf{X}\}$. The equation set becomes a system of linear algebraic equations that can be solved for the unknowns at the boundaries by any standard matrix solver such as Gaussian elimination or LU factorization.

If the boundary conditions in the above example are not properly applied or if internal temperature measurements are included in the analysis, the problem becomes ill-posed, but a solution may still be obtained. If at some boundary points both $u = U$ and $q = Q$ are known while at other boundary points neither is known, the BIE set can still be arranged into the standard form (Martin, Halderman and Dulikravich, 1994). For example, if at two corner points on a quadrilateral cell both $u = U$ and $q = Q$ are known, but at the remaining two corner points neither is known, the BIE set before any rearrangement appears as

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ u_2 \\ U_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{Bmatrix} Q_1 \\ q_2 \\ Q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{6}$$

In order to solve this set, all of the knowns will be collected on the right-hand side, while all of the unknowns are assembled on the left. A simple algebraic manipulation then yields the following set

$$\begin{bmatrix} H_{12} & G_{12} & H_{14} & G_{14} \\ H_{22} & G_{22} & H_{24} & G_{24} \\ H_{32} & G_{32} & H_{34} & G_{34} \\ H_{42} & G_{42} & H_{44} & G_{44} \end{bmatrix} \begin{Bmatrix} u_2 \\ q_2 \\ u_4 \\ q_4 \end{Bmatrix} = \begin{bmatrix} H_{11} & G_{11} & H_{13} & G_{13} \\ H_{21} & G_{21} & H_{23} & G_{23} \\ H_{31} & G_{31} & H_{33} & G_{33} \\ H_{41} & G_{41} & H_{43} & G_{43} \end{bmatrix} \begin{Bmatrix} U_1 \\ Q_1 \\ U_3 \\ Q_3 \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{7}$$

Since the vector on the right-hand side is known, it may be multiplied by its coefficient matrix and added to the vector of known sources, $\{\mathbf{P}\}$, to form a vector of knowns, $\{\mathbf{F}\}$. The left-hand side remains in the form $[\mathbf{A}]\{\mathbf{X}\}$. At first glance, the solution of this set of linear algebraic equations appears straight-forward, but it is not. This equation set is highly singular and most standard matrix solvers will not produce a correct solution.

There exists techniques for dealing with sets of equations that are either singular or nearly singular. These techniques, known as Singular Value Decomposition (SVD) methods, are widely used in solving most linear least squares problems (Press et al. 1992; Throne and Olson, 1994; To, 1994). Any M x N matrix $[\mathbf{A}]$ can be written as the product of an M x N column-orthogonal matrix, $[\mathbf{B}]$, an N x N diagonal matrix $[\mathbf{W}]$ with positive singular values, and the transpose of an N x N orthogonal matrix [C].

$$[A] = [B]\begin{bmatrix} w_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w_N \end{bmatrix}[C] \tag{8}$$

The singular values $w_1, w_2, \ldots, w_N$ are conceptually similar to the eigenvalues of a matrix. For a well conditioned matrix, these values will be roughly of the same order or magnitude. But as the matrix becomes ill-conditioned, that is, more singular, its eigenvalues become more dispersed. Eliminating very small singular values has the effect of removing those algebraic terms that are dominated by noise and round-off error. In order to determine which singular values are eliminated, we must chose a number, δ, as a zero threshold. The choice of δ was based upon the information about the uncertainty in the BEM matrix computation. A good indicator of the accuracy of the matrix formulation, unique to the BEM is the diagonal of the [H] matrix. It may be computed implicitly by assuming a constant temperature and zero heat generation so that

$$h_{ii} = -\sum_{j=1}^{N} h_{ij} \tag{9}$$

This term must equal the internal angle, $\theta_i/2\pi$, between the neighboring boundary panels at the ith node. The singularity threshold, δ, must be, approximately, an order of magnitude larger than the error of this term. This rule, though, only gives us an absolute lower bound. In fact, there is a range of threshold values where the algorithm will produce a correct solution. A choice of δ outside of this range will yield another solution space also satisfying the equation set, but it is wrong. These solutions are characterized as having an oscillatory behavior in the temperature and heat fluxes computed on the unaccessible boundaries. The range of the threshold value, δ, is more thoroughly explained in the section detailing the algorithm's sensitivity to measurement errors in the over-specified boundary data. A SVD algorithm (Press et al., 1992) was used in this work to solve the equation set. Singular values were explicitly zeroed and a solution to the highly singular BIE formulation was obtained. Since the SVD algorithm is capable of solving non-square matrices, the number of unknowns in the equation set need not be the same as the number of equations (Martin and Dulikravich, 1993). Thus, virtually any combination of boundary conditions will yield at least some solution (Okuma and Kukil, 1993). Also, additional equations may be added to the equation set if, for example, temperature or heat flux measurements are known at certain locations within the domain.

ANALYTICAL TEST CASES

The temperature distribution was obtained within an annular domain with an arbitrary heat source distribution, g = g(r,θ), such that the temperature satisfies the Poisson equation

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{1}{r^2}\frac{d^2T}{d\theta^2} + g(r,\theta)/k = 0 \tag{10}$$

The outer circular boundary of the annular domain, at $r = r_b$, is kept at zero temperature, while the inner circular boundary, at $r = r_a$, is thermally insulated. We can obtain an eigenfunction set written in the form of the Helmholtz equation satisfying the homogeneous boundary conditions. After separation of variables and applying the single-valuedness condition, the analytic result for the temperature field in the annular domain is

$$T(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[J_n(\mu_{nm}r) - \frac{J_n(\mu_{nm}r_b)}{Y_n(\mu_{nm}r_b)}Y_n(\mu_{nm}r)\right] \times [A_{nm}\sin n\theta + B_{nm}\cos n\theta] \tag{11}$$

Here, J_n and Y_n are the Bessel functions of integer order n, and μ_{nm} are the roots of the characteristic equation. The Fourier coefficients, A_{nm} and B_{nm}, may be found knowing that the eigenfunctions form an orthogonal set. For example,

$$A_{nm} = \frac{r_a^2}{\pi\mu_{nm}N_{nm}} \times \int_{r_a}^{r_b}\int_0^{2\pi} r\, g(r,\theta)R_{nm}(r)\sin(n\theta)\,dr\,d\theta \tag{12}$$

where $k = 1.0\ W\ m^{-1}\ K^{-1}$ was assumed. If the boundary conditions and the heat generation in the cylindrical domain are axisymmetric, then the analytic solution simplifies to

$$T(r) = \int_0^r \left[\frac{1}{r}\int_0^r g(r)\, r\, dr\right] dr + c_1 \ln r + c_2 \tag{13}$$

RESULTS

A BEM algorithm was developed using the theory discussed in the numerical formulation. The application of the Kirchoff transformation, used to remove the nonlinearity when the coefficient of thermal conductivity varies arbitrarily as a function of temperature, has been verified against a known analytic solution for the heat flow in a finite thin rod (Martin and Dulikravich, 1994a; 1994b).

In order to test our inverse (ill-posed) boundary condition method governed by the Poisson equation, we chose to solve first the forward or well-posed problem in an annular region between two concentric circles (with $r_a = 0.5$ m and $r_b = 1.2$ m) subject to axisymmetric thermal boundary conditions ($T_a = 0.0$ K and $T_b = 0.0$ K) and constant heat generation function, $g(x,y) = 1.0\ W\ m^{-3}$. The coefficient of thermal conductivity was constant, $k = 1.0\ W\ m^{-1}\ K^{-1}$. The analytic solution is easy to obtain from equation (13) indicating that the maximum temperature, $T_{max} = 0.06258$ K, is reached at the radial distance of 0.8244 m.

This solution was then compared to the numerical BEM solution of the same problem. Both inner and outer circular boundaries were discretized with 36 non-clustered, linear, isoparametric flat panels and the annular domain was discretized with 36 x 10 quadrilateral cells. Comparison of analytic and computed radial temperature distributions (Figure 1) and relative percentage errors (Figure 2) demonstrate high accuracy of the BEM.

Next, the boundary conditions supplied to the BEM algorithm were changed to make the problem ill-posed. The outer circular boundary was specified with $T_b = 0.0$ K and a normal temperature derivative, $(dT/dn)_b = -0.3168184$ K m^{-1}, taken from the analytic solution. At the same time nothing was specified on the inner circular boundary. A 72 x 72 BEM matrix set was solved using the SVD matrix solver with a singularity threshold parameter of $\delta = 1.0 \times 10^{-3}$. Figure 2 illustrates the percentage error as a function of radius generated by the BEM for the ill-posed problem. Clearly, the BEM was capable of accurately predicting the unknown boundary values on the inner circular boundary as well as the temperature field in the entire annular domain in a single, non-iterative run without the use of any regularizers or mollifiers. The computing time was less than half of a second on a Cray C-90 computer with a single processor.

Our BEM algorithm was then tested against the complete analytic solution for the same geometry, but with the heat generation taken as a function of both radius, r, and azimuthal angle, θ. Specifically, we used the following expression when computing the coefficients in equation (12).

$$g(r,\theta) = g_{max} \sin\left[\frac{r - r_{in}}{r_{out} - r_{in}}\pi\right]\sin\theta \qquad (14)$$

For the well-posed problem, the outer circular boundary temperature was specified to be $T_b = 0.0$ K, while the inner circular boundary was kept adiabatic, $(dT/dn) = 0.0$ K m^{-1} and $g_{max} = 1.0$ W m^{-3}. Both outer and inner boundaries were discretized with 36 linear panels and the domain was discretized with 36 x 20 quadrilateral cells. The comparison of the well-posed BEM solution to the analytic solution of the Poisson equation for this test case is shown in Figure 3a. The agreement is excellent showing an average error of less than half of a percent.

Next, the outer boundary heat fluxes were taken from analytic solution and supplied as over-specified boundary conditions together with temperature on the outer circular boundary. No boundary conditions were specified on the inner circular boundary. The results of this inverse (ill-posed) boundary condition problem versus the analytic results are shown in Figure 3b. Notice that the largest percentage error found in the domain is less than 0.65% and that the solution errors in both the direct (well-posed) and inverse (ill-posed) problems are very low and nearly identical.

Results With Input Data Noise

The major concern of researchers working on the inverse problems is with the sensitivity of their algorithms to errors in the specified boundary data. In order to verify that our technique did not amplify the input data errors, random Gaussian noise was introduced into the temperature boundary condition supplied to the outer circular boundary. The same annular geometry was used for this purpose and the heat generation was kept at a constant $g(x,y) = 100.0$ W m^{-3}. For the temperature boundary condition on the outer boundary a random real number R between 0.0 and 1.0 was generated using the RANF subroutine on the Cray C-90. Using this value as the normalized probability density function, a randomized temperature boundary condition on the outer circular boundary was determined from the following equation

$$T(\theta) = T_{bc} \pm \sqrt{-2\sigma^2 \ln[R(\theta)]} \qquad (15)$$

Here, $T_{bc} = 0.0$ K is the mean value of the temperature boundary condition on the outer boundary and σ^2 is the variance. The outer circular boundary was also specified with the flux taken from the analytic solution. No boundary conditions were specified on the inner circular boundary. Our BEM program was tested with a variety of variances while computing the unknown temperatures and heat fluxes on the inner circular boundary. Figures 4a, 4b and 4c depict the percent errors both supplied to the program as outer boundary temperatures and those computed by the program as inner boundary temperatures and normal temperature derivatives. These figures serve to prove that our BEM algorithm does not noticeably amplify errors in the input measurement data for the solution of the inverse (ill-posed) boundary condition problem governed by the Poisson equation.

Using numerical experimentation, we formed the following table that should be helpful in chosing the singularity threshold, δ, used in the SVD code.

Case	Variance σ	Threshold δ
a	0.00001	0.1×10^{-5}
b	0.0001	0.5×10^{-5}
c	0.001	0.5×10^{-4}
d	0.01	0.5×10^{-3}
e	0.1	0.5×10^{-2}
f	0.2	0.5×10^{-1}
g	0.5	0.5×10^{-1}

Results With Interior Temperature Measurements

Finally, our approach to solving the inverse (ill-posed) boundary condition problems governed by the Poisson equation was shown to be capable of using internal temperature measurements at isolated points. Given the same annular geometry, only the temperature $T_a = 0.0$ K was specified on the inner circular boundary. No boundary conditions were specified on the outer circular boundary. Instead, analytical values for temperatures at various locations within the domain were used as additional input data. These temperatures ($T = T_{max} = 0.065$ K) corresponded to the analytical solution of the Poisson equation with T_a = 0.0 K, T_a = 0.0 K and g(x,y) = 1.0 W m^{-3} (Figure 1). The values of temperature were specified at the finite number of circumferentially equidistantly spaced points in the annular domain. Figure 5a illustrates the isotherms computed by our inverse (ill-posed) boundary condition BEM algorithm when only four circumferentially equidistantly spaced internal temperature values were. The errors in the predicted temperature on the inner circular boundary are significant. This is understandable since the resulting BEM equation set contained 76 equations and 108 unknowns. The isotherms in Figures 5b and 5c result from using 6 and 9 equidistantly spaced internal temperature points, respectively. From these figures one notices that our algorithm produces very good results when at least 9 thermocouples are used at points within the annular domain and temperature is given on only one boundary.

CONCLUSIONS

Our boundary element or Green's function method has been shown to provide stable and accurate solutions to several simple ill-posed problems of the Poisson equation where the boundary conditions were unknown on certain boundaries. The algorithm is non-iterative because it uses the BEM. Our algorithm has been shown to be robust and fast. Its accuracy has been proven against several two-dimensional heat conduction analytic solutions. Furthermore, our technique does not iterate to minimize a global function based upon the residual between the overspecified and computed boundary values. The magnification of errors in measurement data, the need for mollifiers to smooth the intermediate predictions and the influence of regularizers on the physics of the problem have been eliminated because the procedure is non-iterative. Our method can be readily extended to the solution of three-dimensional inverse (ill-posed) boundary condition problems governed by the Poisson equation.

ACKNOWLEDGMENTS

Authors are grateful for the remote access to NASA Ames Research Center NAS facility Cray C-90 computer granted by NASA Lewis Research Center. Post processing was performed on the equipment donated by Apple Computer. Mr. Norman F. Foster's help with the graphics and Mr. Sean R. Lynn's help with proofreading are gratefully acknowledged. The lead author is grateful for the NASA Graduate Traineeship received from the NASA-Pennsylviaia State University Propulsion Engineering Research Center.

REFERENCES

Backus, B., and Gilbert, F., 1970, "Uniqueness in the Inversion of Inaccurate Gross Earth Data", *Phil. Trans. R. Soc. London*, Vol. 266, pp.123-192.

Beck, J. V., Blackwell, B., and St. Clair, C. R. Jr., 1985, "*Inverse Heat Conduction: Ill-Posed Problems*", Wiley-Interscience, New York.

Brebbia, C. A., 1978, "*The Boundary Element Method for Engineer"s*, John Wiley & Sons, New York.

Brebbia, C. A., and Dominguez, J., 1989, "*Boundary Elements, An Introductory Course*", McGraw-Hill Book Company, New York..

Dulikravich, G. S., and Martin, T. J., 1994ba "Inverse Problems and Design in Heat Conduction", Proceedings of 2nd International Symposium on Inverse Problems in Engineering, Nov. 2-4. 1994, Editors: Ed. H. D. Bui, M. Tanaka, M. Bonnet, H. Maigre, E. Luzzato and M. Reynier, Paris, France, November 2-4, 1994, A. A. Balkema, Rotterdam, 1994, pp. 13-20.

Dulikravich, G. S., and Martin, T. J., 1994b, "Inverse Design of Super-Elliptic Cooling Passages in Coated Turbine Blade Airfoils", *AIAA Journal of Thermophysics and Heat Transfer*, Vol. 8, No. 2, pp. 288-294.

Hensel, E., and Hills, R. G., 1986, "An Initial Value Approach to the Inverse Heat Conduction Problem", *Journal of Heat Transfer*, Vol. 108, pp. 248-256.

Kagawa, Y., Sun, Y.-H., and Matsumoto, O., 1995, "Inverse Solution of Poisson Equation Using DRM Boundary Element Model - Identification of Space Charge Distribution", *Inverse Problems in Eng.*, Vol. 1, No. 3.

Lanczos, C. 1961, "*Linear Differential Operators*", VanNostrand, London.

Martin, T. J., and Dulikravich, G. S., 1993, "A Direct Approach to Finding Unknown Boundary Conditions in Steady Heat Conduction", Proc. of 5th Annual Thermal and Fluids Workshop, Lewis Research Center, Ohio, Aug. 16-20, 1993, NASA CP-10122, pp. 137-149.

Martin, T. J., and Dulikravich, G. S., 1994a, "Inverse Determination of Temperatures and Heat Fluxes on Inaccessible Surfaces", in *Boundary Element Technology IX*, Computational Mechanics Publications, Southampton, Editors: C. A. Brebbia and A. Kassab, pp. 69-76.

Martin, T, J., and G. S. Dulikravich, G. S., 1994b, "Finding Unknown Surface Temperatures and Heat Fluxes in Steady Heat Conduction", 4th InterSociety Conference on Thermal Phenomena in Electronic Systems (I-THERM IV), May 4-7, 1984, Washington, D.C., Editors: A. Ortega and D. Agonafer, pp. 214-221; also to appear in *IEEE Transactions of Components, Packaging and Manufacturing Technology (CPMT) Society - Series A*, June 1995.

Martin, T. J., Halderman, J., and Dulikravich, G. S., 1994, "An Inverse Method for Finding Unknown Surface Tractions and Deformations in Elastostatics", ASME Winter Annual Meeting, Chicago, IL, November 6-11, 1994, Editors: S. Saigal and L. G. Olson, ASME AMD-Vol. 186, pp. 57-66.; also to appear in *Computers and Structures*, 1995.

Murio, D. A., 1993, "The Mollification Method and the Numerical Solution of Ill-Posed Problems", John Wiley & Sons, Inc., New York.

Okuma, M., and Kukil, S., 1993, "Correction of Finite Element Models Using Experimental Modal Data for Vibration Analysis", in *Inverse Problems in Mechanics*, Editor: S. Kubo, Atlanta Technology Publications, Atlanta, GA, pp. 204-211.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P., 1992, "*Numerical Recipes in FORTRAN*", Second Edition, Cambridge University Press.

To, W. M., 1994, "Application of Singular Value Decomposition to Direct Matrix Update", *AIAA Journal*, Vol. 32, No. 10, pp. 2124-2126.

Throne, R. D., and Olson, L. G., 1994, "A Generalized Eigensystem Approach to the Inverse Problem of Electrocardiography", *IEEE Transact. on Biomedical Eng.*, Vol. 41, no. 6, pp. 1-9.

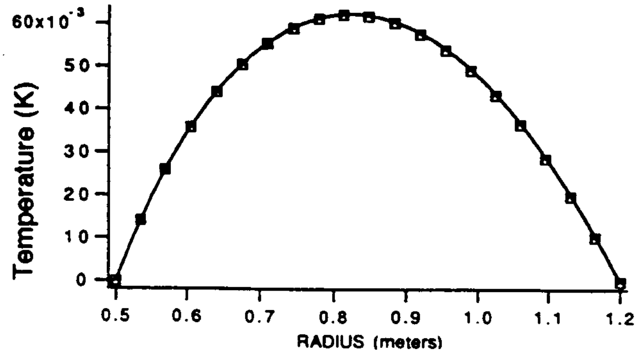

Figure 1. Radial temperature distribution computed by the well-posed BEM (squares) compared with the analytic results (line) of the Poisson equation on an annular domain with r_a = 0.5 m, r_b = 1.2 m, g(x,y) = 1.0 W m^{-3}, T_a = T_b = 0.0 K.

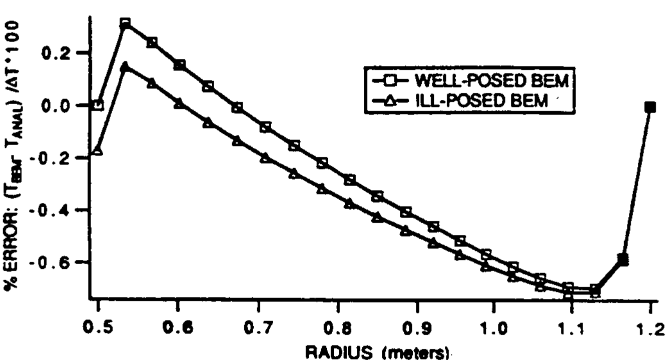

Figure 2. Comparison of errors in the computed radial temperature distribution compared with the analytic results of the Poisson equation on an annular domain with r_a = 0.5 m, r_b = 1.2 m, g(x,y) = 1.0 W m^{-3}, T_a = T_b = 0.0 K.
a) Error in numerical results obtained with our direct (well-posed) BEM with T_a = T_b = 0.0 K,
b) Error in numerical results obtained with our inverse (ill-posed) BEM with T_a = $(dT/dn)_a$ = unknown, T_b = 0.0 K, $(dT/dn)_b$ = -0.3168184 K m^{-1} (from the analytic solution). The temperature differences were normalized with ΔT = 0.0625 K.

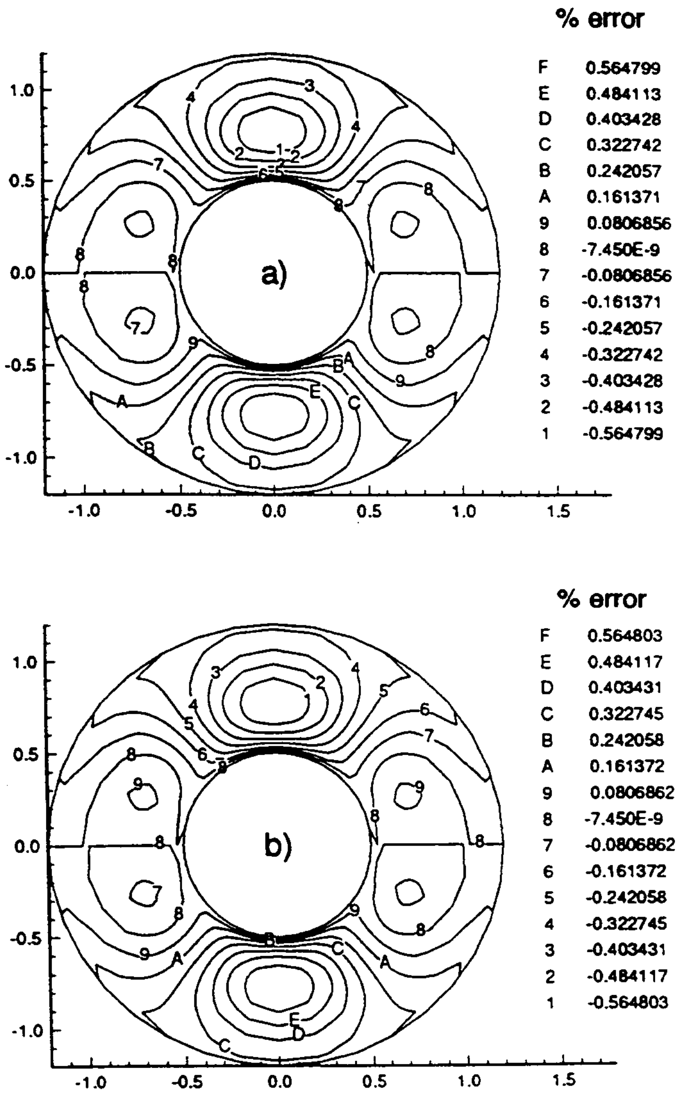

Figure 3. Contours of constant error levels in computed temperatures compared with the analytic results of the Poisson equation on an annular domain with r_a = 0.5 m, r_b = 1.2 m, g(x,y) = sin{[(r - r_a)/(r_b - r_a)]π} sinθ W m^{-3}, $(dT/dn)_a$ = 0.0 K m^{-1}, T_b = 0.0 K.
a) Error in numerical results obtained with our direct (well-posed) BEM with $(dT/dn)_a$ = 0.0 K m^{-1} and T_b = 0.0 K.
b) Error in numerical results obtained with our inverse (ill-posed) BEM with T_a = $(dT/dn)_a$ = unknown, T_b = 0.0 K, $(dT/dn)_b$ = from the analytic solution. The temperature differences were normalized with ΔT = 0.2 K.

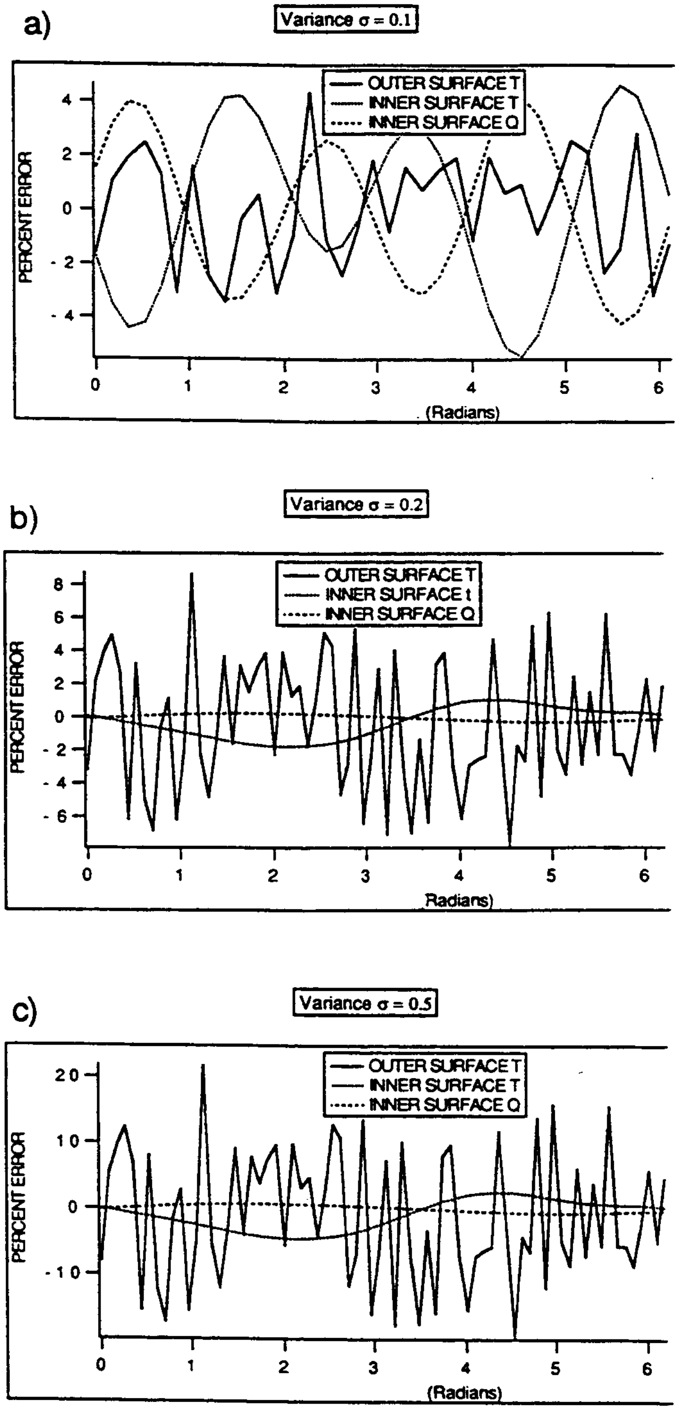

Figure 4. Specified (input) temperature percentage errors on the outer circular boundary and computed (output) percentage errors in temperature and heat flux on the inner circular boundary for $r_a = 0.5$ m, $r_b = 1.2$ m, $g(x,y) = 100.0$ W m^{-3}, $T_a = (dT/dn)_a$ = unknown, $T_b = 0.0$ K plus/minus a randomized error, $(dT/dn)_b = -31.68184$ K m^{-1} (from the analytic solution). The input temperature variances were: a) $\sigma = 0.1$, b) $\sigma = 0.2$, c) $\sigma = 0.5$.

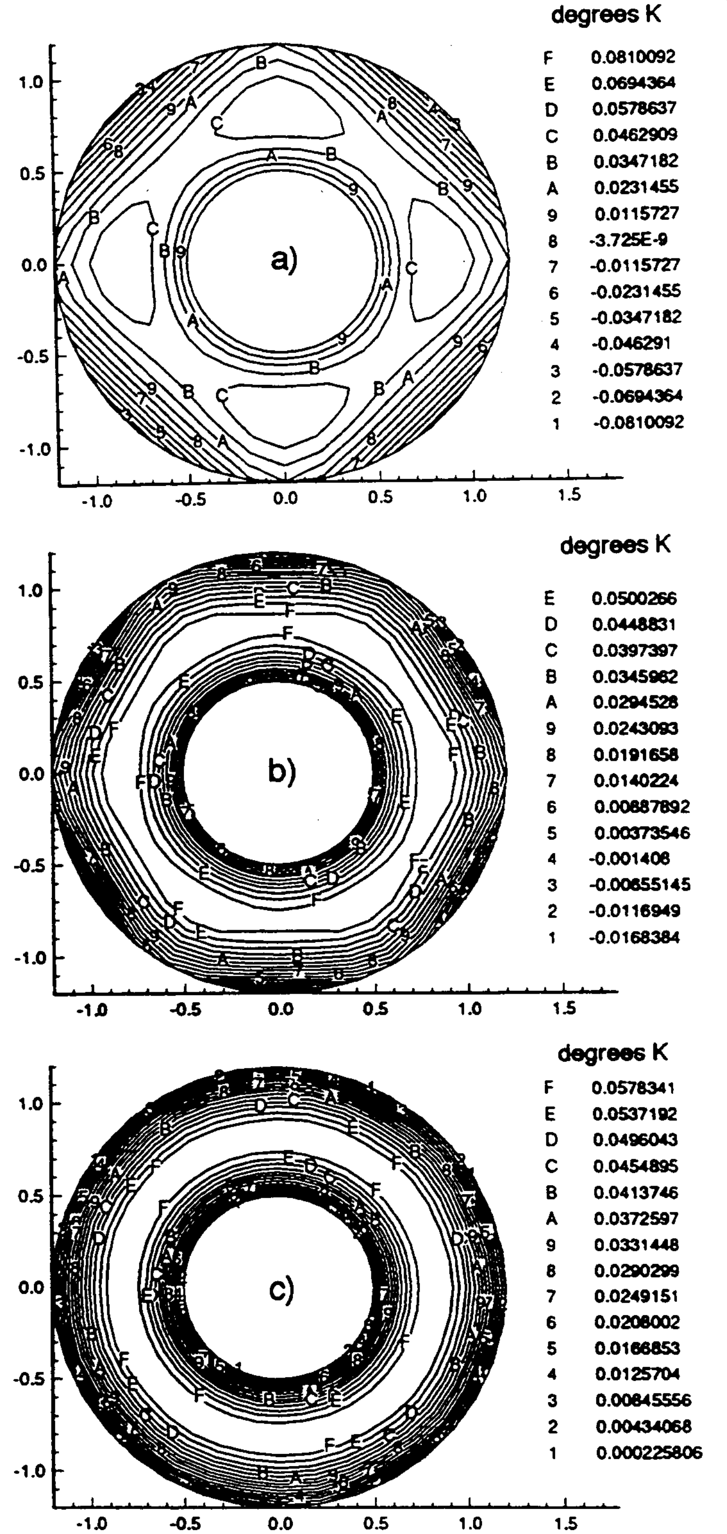

Figure 5. Isotherms computed in the annular region with our inverse (ill-posed) BEM with $g(x,y) = g(x,y) = 1.0$ W m^{-3}, T_a = unknown, $T_b = 0.0$ K and interior temperature measurements provided at: a) four, b) six, and c) nine circumferentially equidistantly spaced isolated points. The correct answer should be $T_a = 0.0$ K.