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Inverse Determination of Temperatures and Heat Fluxes on Inaccessible Surfaces

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ABSTRACT

A method for determining thermal boundary conditions on surfaces of thermally conducting solids where such quantities were unknown has been developed and demonstrated. The method uses a non-iterative or direct approach in solving steady inverse heat conduction problems. Given any over-specified thermal boundary conditions such as a combination of temperature and heat flux on any portion of the surface, the temperature field within the object and any unknown thermal boundary conditions will be computed on surfaces where thermal boundary values are unavailable. A steady-state non-linear heat conduction BEM algorithm has been developed in two and three dimensions and was tested on several simple geometries where the analytic solutions are known. The accuracy and reliability of this technique is very good but tends to deteriorate when the known surface conditions are only slightly over-specified and far from the inaccessible surface. In addition, the algorithm was found not to be overly sensitive to measurement errors in the boundary values.

INTRODUCTION

The objective of the steady-state inverse heat conduction problem (IHCP) is to deduce temperatures and heat fluxes on a portion of an object's surface where such information is unknown [Beck et al. 1985, Hensel 1986]. In many instances it is impossible to place thermal sensors and take measurements on a particular surface of a conducting solid due to the inaccessibility or severity of the environment on that surface. A characteristic of most of the inverse techniques of the past is that they tend to produce temporal oscillations in the unknown surface thermal condition estimates that are larger than the temporal oscillations in the over-specified thermal data as it propagates through the solid [Hills and Hensel 1986]. In other words, the random noise due to round off errors tends to magnify as the solution proceeds especially as the distance between the surface and the over-specified information increases. A number of authors have presented various smoothing techniques for reducing this error growth, but the effect of these operations on the accuracy of the solution is not easy to evaluate [Murio 1993].

THEORY

The method presented in this paper does not utilize any artificial smoothing technique and is not limited to transient or one-dimensional problems. This approach is non-iterative and has been shown [Martin and Dulikravich 1993] to compute meaningful and accurate thermal fields in a single analysis using a straight-forward modification to the boundary element method. When using the BEM in solving the IHCP, the unknown thermal boundary values may be deduced from additional temperature or heat flux measurements made at arbitrary locations within the solid or on some other surface of the solid. For well-posed steady-state heat conduction analysis, either temperatures, T, or heat fluxes, Q, are specified everywhere on the surface of the solid such that at each point on the boundary one of these quantities is known while the other is unknown. In the BEM solution to the IHCP, both T and Q must be specified on any portions of the solid's surface, while both T and Q are unknown on other portions of the surface. Elsewhere on the solid's surface, a single boundary condition should be applied as either T or Q. The surface section where both T and Q are specified is called the over-specified boundary and is necessary in the IHCP's solution methodology. This leads to a highly singular BEM matrix problem of the type $[\mathbf{H}]\ \mathbf{T} = [\mathbf{G}]\ \mathbf{Q}$, where $\mathbf{T} = \{T_1, T_2, ..., T_N\}$ and $\mathbf{Q} = \{Q_1, Q_2, ..., Q_N\}$. If known temperature measurements are supplied to points within the solid, one additional equation for each data point may be added to the equation set. After the $[\mathbf{H}]$ and $[\mathbf{G}]$ are formed, all boundary conditions are applied and a set of linear algebraic equations, $[\mathbf{A}]\ \{\mathbf{X}\} = \{\mathbf{F}\}$, is constructed. Known or specified values of T and Q are assembled on the right-hand-side of the equation set and are multiplied by their respective $[\mathbf{H}]$ or $[\mathbf{G}]$ matrix row thus forming the vector of knowns, $\{\mathbf{F}\}$. All unknown T and Q are assembled on the left-hand-side of the equation set and are represented by a coefficient matrix $[\mathbf{A}]$ multiplying a vector of unknown quantities, $\{\mathbf{X}\}$, which can be solved accurately using a special numerical algorithm for singular matrices.

Steady-state heat conduction in a nonhomogeneous, isotropic medium with a temperature-dependent coefficient of thermal conductivity, $\lambda(T)$, is governed by the following non-linear elliptic partial differential equation $\nabla \bullet \left(\lambda(T)\ \nabla T \right) = 0$ where T is the temperature. This equation represents a boundary value problem having essential boundary conditions, T_0, and natural boundary conditions, Q_0. Equation (1) can be linearized by the application of the classical Kirchoff transformation which defines the heat function, Θ, as

$$\Theta = \int_0^T \frac{\lambda(T)}{\lambda_o}\, dT \qquad (1)$$

Here, λ_o is a reference conductivity. Utilizing this transformation, equation (1) can be transformed into Laplace's equation and solved for the heat function, Θ, instead of the temperature, T. Results obtained for the heat function must be transformed back into temperatures using the inverse of the transformation given in equation (1). The Laplace's equation can be accurately and efficiently solved using the BEM [Brebbia and Dominguez 1989]. For each two-dimensional test case, surfaces were discretised into straight segments. The variation of Θ and Q along each panel was linear. In addition, the program allowed for two values of flux at each node in order to allow for proper corner treatment. The surface element integrals were performed with four-point Gaussian quadrature.

Whenever the surface element integration included a singularity, that is, when the observation node was included at one of the end nodes of the surface segment, an analytical expression was used for the surface integral. The surfaces of each three-dimensional test case were discretised with isoparametric quadrilateral panels. The integration over each surface panel was performed with three-point Gaussian quadrature. Whenever the surface panel integral included a singularity at one of the quadrilateral's vertices, a localized transformation was performed to eliminate the singularity and the order of the Gaussian quadrature integration was increased to five-point.

RESULTS AND DISCUSSION

Computer programs for two- and three-dimensional steady IHCP have been developed based on the theory discussed in the previous section.

IHCP for a Square Plate

The accuracy of the BEM as a solution to the IHCP was verified for a solid square plate (6.0 m x 6 m) with λ = constant. The top and bottom boundaries were specified to be adiabatic while the left boundary of the plate was over-specified with T = 300 K and a Q = -50 W/m^2. Both T and Q were assumed to be unknown on the right boundary. The plate circumference was discretized with 12 panels. The BEM was successful in computing T and Q on the right boundary to within 0.0001 % and 0.000006 % error, respectively.

Study of the IHCP for an Annular Disk

The behavior of the algorithm for various combinations of boundary conditions was documented for the two-dimensional, steady-state heat conduction in an annular solid disk [Martin 1993]. The outer radius of the disk was 1.2 m with T = 100 K and the centrally located circular hole had a radius of 0.5 m with T = 50 K and λ = constant. The analytic solution is T (r) = 89.59 + 57.11 ln r. Correspondingly, Q_{out} = -47.59 W/m^2 and Q_{in} = 114.22 W/m^2. Four variations to this problem were tested [Martin and Dulikravich 1993] and compared to the analytic solution. Both the outer and inner surfaces of the annular disk were discretised with 36 surface panels.

Test 1 The entire outer boundary was over-specified with T and Q, while both T and Q were assumed to be unknown on the inner boundary. For this test case, the solution set of 72 equations included 72 known values given as boundary conditions on the outer surface and 72 unknowns on the inner boundary. The BEM computed the temperature field within the annular solid in addition to the unknown temperatures and heat fluxes on the inner boundary. Figure 1a shows the computed temperature contours and also includes the position and type of boundary conditions used at each node. Figure 2 shows the relative percentage error in T on the inner boundary for each test as a function of the circumferential angle in radians. Figure 3 is similar to figure 2 except that it gives the relative percentage error in Q on the inner boundary. Notice that this test case had an almost perfectly symmetric result with an average error of less than 0.0005% in T and uniformly biased error of about 0.068% in Q. The bias in Q was probably due to the fact that flat panels were used to model the circular geometry.

Test 2 Temperature boundary conditions were specified everywhere on the outer boundary. The additional heat flux boundary conditions were over-specified in the first and third quadrants of the outer boundary only. The BEM solution set had 54 knowns, 90 unknowns and 72 equations. Figure 1b illustrates that this configuration produced nearly symmetric isotherms. The T on the inner boundary was somewhat oscillatory, but averaged only a 0.85% error (Fig. 2). The Q on the inner boundary was also oscillatory and averaged an error of -3.5% (Fig. 3).

Test 3 Temperature was specified over the entire outer boundary, while Q was over-specified only on the upper half of the outer boundary. As in Test 2, the BEM solution set contained 54 unknowns, 90 unknowns and 72 equations. The isotherms (Fig. 1c) were asymmetric about the x-axis, but were very nearly symmetric about the y-axis. The greatest error in the temperature field occurred at the center of the bottom half of the annular solid region (Figs. 2 and 3).

Test 4. This test case is similar to Test 2, except that Q was over-specified in the first quadrant of the outer boundary only. The BEM solution set contained 45 knowns, 99 unknowns and 72 equations. Figure 1d illustrates the temperature contours within the solid disk. The error in the temperature field obviously worsens as the distance from the over-specified data increases (Figs. 2 and 3).

Study of the Sensitivity of the IHCP to Measurement Errors

Various degrees of errors were intentionally introduced to the over-specified boundary conditions of the annular disk described in the previous section. The sensitivity of the computed T and Q values on the inner circular boundary to errors in the over-specified boundary conditions on the outer boundary was evaluated by plotting the standard deviation of the outer boundary T or Q values versus the standard deviation of the computed inner boundary T and Q values. The results of this analysis (Figs. 4 and 5) illustrate that the standard deviation of the Q output was always greater than those of T output. In addition, the output (inner surface) errors stayed about the same for both T and Q until a standard deviation of about 0.1. Further increase in the input errors in the outer boundary data had a linear relationship to the increase in the errors in the output values. This demonstrates that the BEM solution to the IHCP is not overly sensitive to errors in the measurement data.

IHCP for a Centered Spherical Cavity within a Sphere

A three-dimensional BEM version of the inverse heat conduction technique described in the previous sections has been developed and demonstrated against the known analytic solution of a concentric spherical cavity within a sphere. The radii of the outer and inner surface were 1.0 m and 0.5 m, respectively, with λ = constant. The analytic solution for T_{out} = 100 K and T_{in} = 50 K is $T(r) = 150 - 50/r$. Thus, $Q_{out} = -50$ W/m^2 and $Q_{in} = 200$ W/m^2.

This inverse heat conduction problem using the BEM algorithm was tested against this analytic solution. The outer surface of the sphere was over-specified with T_{out} = 100 K and $Q_{out} = -50$ W/m^2. Each outer and inner surface was discretised with 64 isoparametric, quadrilateral surface panels (Fig. 6). The BEM set contained 144 knowns and the same number of unknowns. The results showed that the BEM averaged an error of about 3.1% in T and 4.2% in Q. This discrepancy was attributed to the fact that flat quadrilateral panels were used to model the spherical geometry. In particular, most of the error occurred near the poles where the surface panels were nearly triangular in shape.

Verification of the Nonlinear BEM for a Rectangular Plate

The accuracy of the boundary element analysis program for nonlinear heat conduction in a 1.0 m long by 0.1 m wide rectangular plate was verified. The plate was discretized with 22 linear surface panels each 0.1 m in length. The two vertical sides were kept adiabatic ($Q_0 = 0$) and the two horizontal sides were subject to different uniform temperatures ($T_1 = 100$ K and $T_2 = 0$ K) with $\lambda(T) = \lambda_o (1 + CT)$ where $\lambda_o = 1.0$ W/m K. The analytic solution [Chapman 1960] is

$$\frac{C}{2}T^2 + T = \left(T_1 + \frac{C}{2}T_1^2\right) - \left(1+\frac{C}{2}(T_1 + T_2)\right)\frac{(z - z_1)}{(z_2 - z_1)}(T_1 - T_2) \qquad (2)$$

The BEM computed results for various degrees of non-linearity given by the parameter C. These compared very well to the analytic solution, averaging an error of less than 0.5 % (Fig. 7).

CONCLUSIONS

The BEM algorithm developed for the two- and three-dimensional steady-state IHCP is capable of non-iteratively computing temperature and heat flux boundary conditions on boundaries of a conducting solid where such quantities were originally unknown. The results presented herein indicate that this direct solution method is an accurate, robust and reliable technique that takes only seconds of CPU time on any typical mainframe, workstation or advanced personal computer. The results obtained were found to be more accurate when one or both of the following conditions were met: a) a greater amount of over-specified data was provided, b) the over-specified data locations were in close proximity to the locations of the unknown boundary conditions. In addition, this solution technique was not overly sensitive to measurement errors in the boundary conditions supplied to the program.

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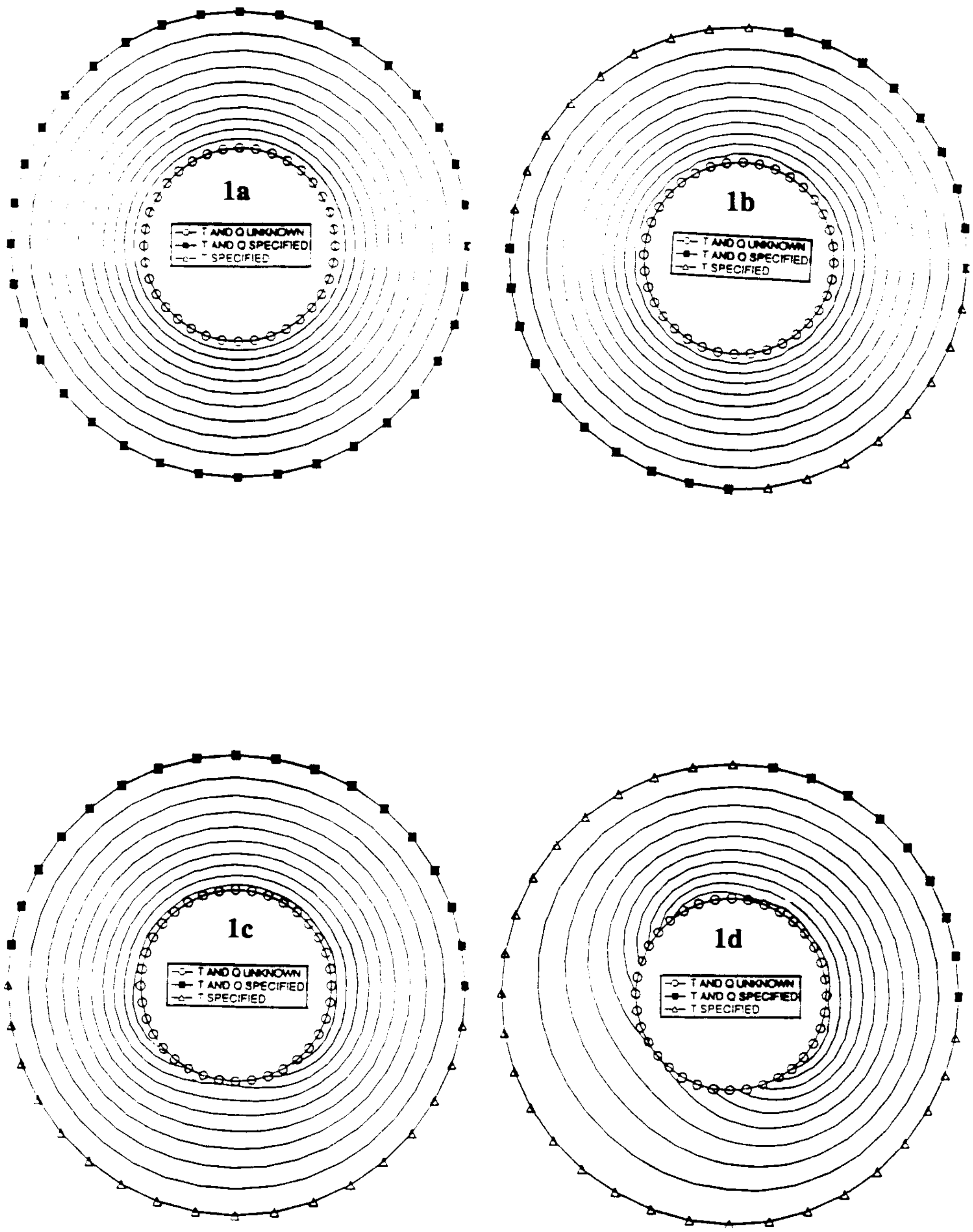

Figure 1: Geometry of the BEM nodes on the outer and inner boundaries, boundary condition types and isotherms computed with the BEM for the four annular disk test cases.

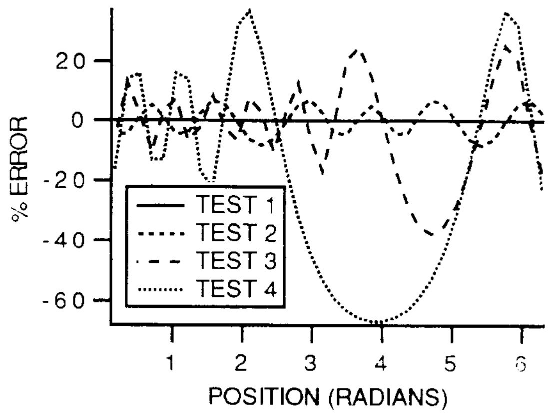

Figure 2: Relative percentage errors (BEM versus analytic solution) of the inner boundary temperatures for the four annular disk test cases.

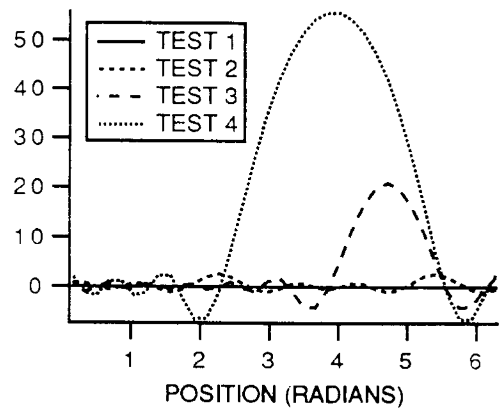

Figure 3: Relative percentage errors (BEM versus analytic solution) of the inner boundary heat fluxes for the four annular disk test cases.

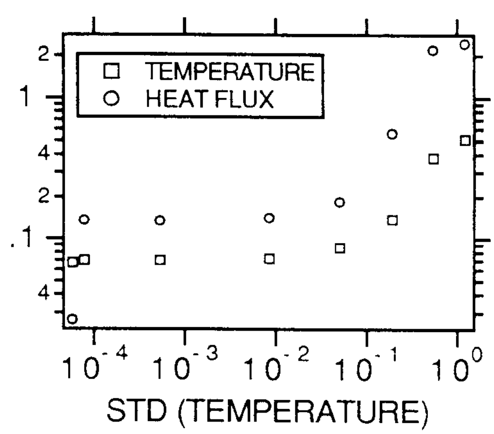

Figure 4. Standard deviation of the inner surface temperatures and heat fluxes due to errors in the temperature data on the outer surface.

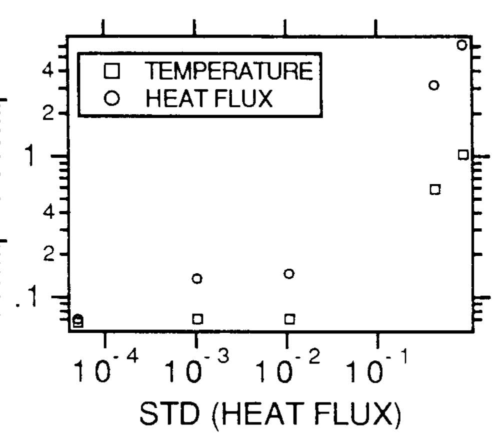

Figure 5: Standard deviation of the inner surface temperatures and heat fluxes due to errors in the heat flux data on the outer surface.

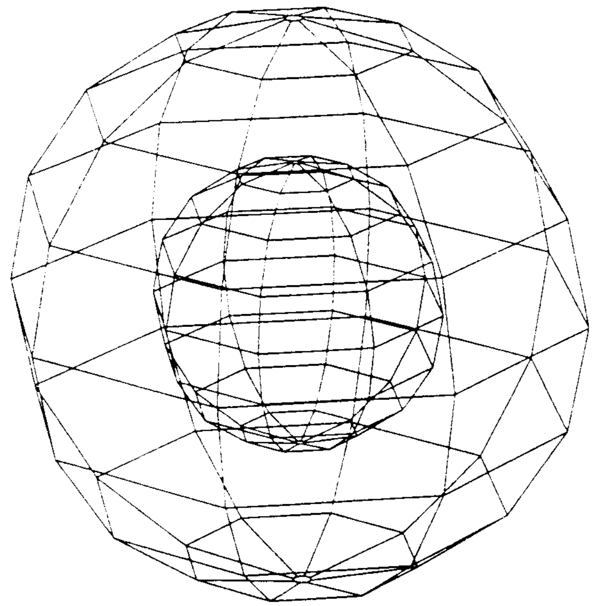

Figure 6: Geometry and discretization of the sphere within a sphere configuration.

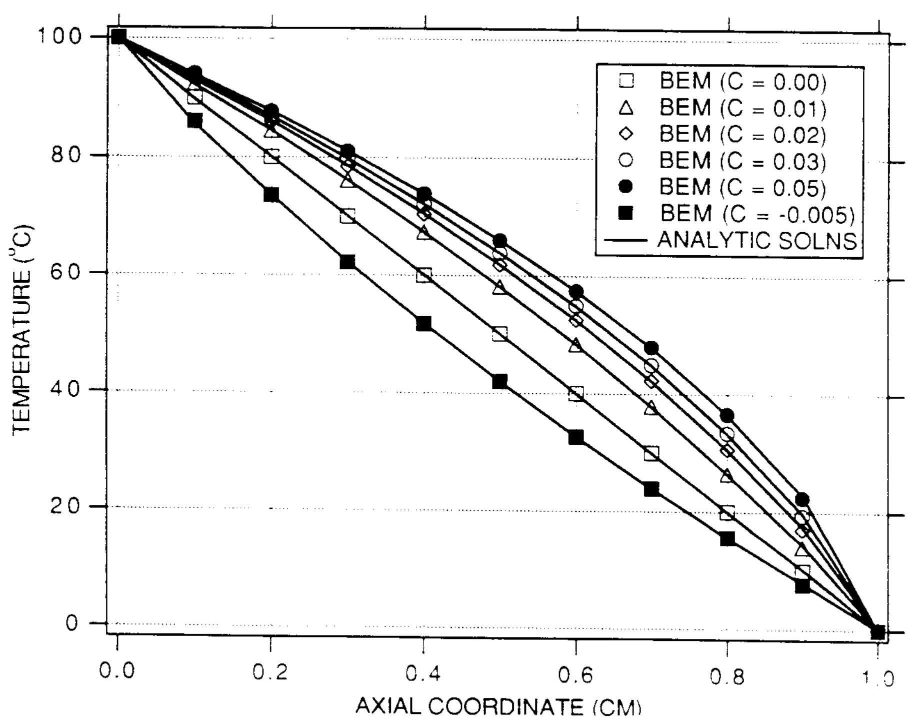

Figure 7: Comparison of the two-dimensional, nonlinear BEM with the one-dimensional analytic solution of a finite thin rod for various degrees of nonlinearity.