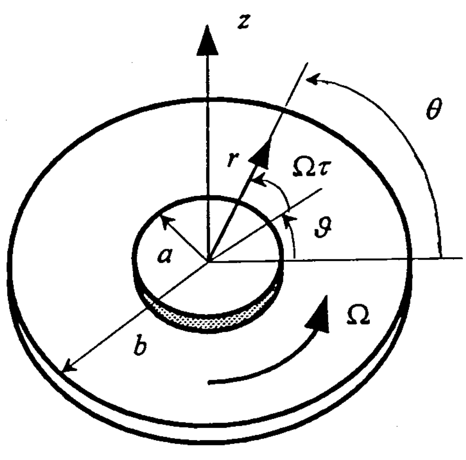
ISSN 1565-1339 VOLUME 1, NO. 4, 2000

International Journal of Nonlinear Sciences and Numerical Simulation

Editor-in-Chief: Ji-Huan He
Co-Editors: O.S. Limarchenko and R.S. Mackay



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Non-Reflective Boundary Conditions for a Consistent Model of Axisymmetric Electro-Magneto-Hydrodynamic Flows*

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(Received June 8, 2000)

Abstract

In this paper, the non-reflective boundary conditions for the axisymmetric electro-magneto-hydrodynamic (EMHD) flows have been derived. The electro-magneto-hydrodynamics (EMHD) deals with the motion of electrically conducting incompressible fluids under the combined influence of externally applied and internally generated electric and magnetic fields. A consistent axisymmetric EMHD flow model with linear constitutive relations and artificial compressibility was expressed in cylindrical coordinates. After some simplifications, the resulting EMHD system comprised of modified Maxwell equations for the electro-magnetic fields and modified Navier-Stokes equations for the flow-field, was transformed to a characteristic form, and the non-reflective boundary conditions were derived. The results show the strong mutual interactions between the axisymmetric flow-field and the electro-magnetic fields. The limiting cases, including the conventional axisymmetric flow-field model and the electro-magnetic field model in vacuum, are recoverable from these results.

Nomenclature

$\underline{B} = \mu_0(\underline{H} + \underline{M})$	magnetic flux density, kg A^{-1} s^{-2}	$\underline{E}$	electric field intensity, kg m s^{-3} A^{-1}
c_p	specific heat at constant pressure, K^{-1} m^2 s^{-2}	$\hat{\underline{E}} = \underline{E} + \underline{v} \times \underline{B}$	electromotive intensity, kg m s^{-3} A^{-1}
$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$	electric displacement vector, A s m^{-2}	$\underline{f}$	mechanical body force per unit mass, m s^{-2}
$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$	material derivative, s^{-1}	$\underline{H}$	magnetic field intensity, A m^{-1}

* this paper was presented in the Proceedings of Symposium on Rheology and Fluid Mechanics of Nonlinear Materials, ASME International Mechanical Engineering Congress & Exposition, November 14-19, 1999, Nashville, Tennessee

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$\underline{J} = \underline{\hat{J}} + q_e \underline{v}$	electric current density, A m^{-2}
$\underline{\hat{J}}$	electric conduction current, A m^{-2}
$\underline{M}$	total magnetization per unit volume, A m^{-1}
$\underline{\hat{M}} = \underline{M} + \underline{v} \times \underline{P}$	magnetomotive intensity, A m^{-1}
p	pressure, kg m^{-1} s^{-2}
$\tilde{p} = p + \frac{\varepsilon_p(\underline{\hat{E}} \cdot \underline{\hat{E}})}{2} + \frac{(\underline{B} \cdot \underline{B})}{2\mu_m}$	modified pressure, kg m^{-1} s^{-2}
$\underline{P}$	polarization per unit volume, A s m^{-2}
q_e	electric charge density, A s m^{-3}
$\underline{q}$	conduction heat flux, kg s^{-3}
Q_h	heat source per unit volume, kg m^{-1} s^{-3}
t	time, s
$\underline{\underline{t}}$	Cauchy stress tensor, kg m^{-1} s^{-2}
T	absolute temperature, K
$\underline{v}$	fluid velocity, m s^{-1}

__Greek Symbols__

β	Chorin's artificial compressibility, kg m^{-1} s^{-2}
ε	electric permittivity of fluid, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_0 = 8.854 \times 10^{-12}$	electric permittivity of vacuum, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_p = \varepsilon - \varepsilon_0$	polarization electric permittivity, kg^{-1} m^{-3} s^4 A^2
$\varepsilon_r = \varepsilon / \varepsilon_0$	relative electric permittivity of fluid
η	coefficient of fluid viscosity, kg m^{-1} s^{-1}
κ	thermal conductivity coefficient, kg m s^{-3} K^{-1}
κ_E	electro-thermal conductivity coefficient, m^{-1} A
ρ	fluid density, kg m^{-3}
σ	electric conductivity coefficient, kg^{-1} m^{-3} s^3 A^2
σ_T	thermo-electric conductivity coefficient, m^{-1} A K^{-1}
μ	magnetic permeability of fluid, kg m A^{-2} s^{-2}
$\mu_0 = 4\pi \times 10^{-7}$	magnetic permeability of vacuum, kg m A^{-2} s^{-2}
$\mu_m = \frac{\mu_0 \mu}{\mu - \mu_0}$	magnetization magnetic permeability, kg m A^{-2} s^{-2}

1. Introduction

Electro-magneto-hydrodynamics (EMHD) is an interdisciplinary study of electro-magnetics and fluid dynamics. It deals with the flow of polarizable and magnetizable incompressible fluids under the combined effects of electric and magnetic fields. The conventional electro-hydrodynamics (EHD) and magneto-hydrodynamics (MHD) are two extreme cases of EMHD. Recently, a series of rigorous theoretical continuum mechanics treatments of EMHD flows (Eringen and Maugin, 1990a, 1990b; Lakhtakia, 1993; Dulikravich and Lynn, 1995a, 1995b, 1997a, 1997b; Dulikravich and Jing, 1996, 1997; Dulikravich, 1998; Ko and Dulikravich, 1998) have been developed. Especially, Eringen and Maugin (1990a; 1990b) laid the firm foundations of the EMHD theory and developed the most complete and robust model for the balance laws and constitutive equations. Following their general formulation, Ko and Dulikravich (1998) developed a fully consistent non-linear multi-dimensional EMHD model.

A complete set of boundary conditions is required in order to perform numerical simulation of EMHD flows. Usually the no-slip conditions and jump conditions are imposed at the solid walls, while non-reflective (for incoming waves) or characteristic (for outgoing waves) boundary conditions are used at the open boundaries. Shankar et al. (1989) and Shang (1991) have performed numerical simulations of electro-magnetic fields with fluid flow, but they did not include the effects of fluid polarization and magnetization. Although the effects of polarization and magnetization was treated inconsistently, Dulikravich and Jing (1996) performed an analytical formulation of the EMHD flow model and formulated the procedures to define open boundary conditions (Dulikravich and Jing, 1997). Based on their recently developed fully consistent non-linear EMHD theory, Ko and Dulikravich (1998) derived non-reflective boundary conditions for two-dimensional planar EMHD flows (Ko and

Dulikravich, 1999). Like the numerous studies on the open boundary conditions, such as for MHD flows by Sun et al. (1995) and for acoustics by Reitsma et al. (1993), this formulation was an extension of Thompson's (1987, 1990) method for the general hyperbolic system of equations.

This paper presents a formulation of the non-reflective boundary conditions at the open boundaries for an axisymmetric EMHD flow model, thus complementing the study for the two-dimensional planar EMHD case (Ko and Dulikravich, 1999). For this purpose, the consistent linear EMHD model will be investigated. With the help of certain modifications and simplifications, the characteristic boundary conditions will be derived from the characteristic form of the governing equations, by following Thompson's approach. This work, like its planar flow counterpart, will supercede the results of Dulikravich (1998) based on an inconsistent EMHD model.

2. Governing Equations of EMHD Flows

A complete set of balance laws governing the general electro-magneto-gas-dynamic (EMGD) flows has been derived by Eringen and Maugin (1990a; 1990b) using a continuum mechanical approach. The corresponding set of electro-magneto-hydrodynamic (EMHD) flow equations was formulated by Ko and Dulikravich (1998). It consists of modified Maxwell's equations governing electro-magnetism in a moving fluid, the modified Navier-Stokes equations governing heat and fluid flow under the influence of electric and magnetic fields, and constitutive equations (Wineman and Rajagopal, 1995) describing behavior of the fluid.

It can be shown (Dulikravich, 1998; Ko and Dulikravich, 1999) that a vector operator form of the set of equations governing EMHD flows can be written in the rationalized MKS system as a combination of the reduced electro-magnetic subsystem

$$\varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \nabla \times \underline{H} - \underline{J} - \frac{\partial \underline{P}}{\partial t}, \qquad (1)$$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}, \qquad (2)$$

$$\frac{\partial q_e}{\partial t} + \nabla \cdot \underline{J} = 0, \qquad (3)$$

and the thermo-mechanical subsystem

$$\nabla \cdot \underline{v} = 0, \qquad (4)$$

$$\rho \frac{D\underline{v}}{Dt} = \nabla \cdot \underline{\underline{t}} + \rho \underline{f} + q_e \underline{E} + \underline{J} \times \underline{B} + (\nabla \underline{E}) \cdot \underline{P} + (\nabla \underline{B}) \cdot \underline{M} + \nabla \cdot (\underline{v}(\underline{P} \times \underline{B})) + \frac{\partial}{\partial t}(\underline{P} \times \underline{B}), \qquad (5)$$

$$\rho c_p \frac{DT}{Dt} = Q_h + \nabla \cdot \underline{q} + \hat{\underline{J}} \cdot \hat{\underline{E}} + \hat{\underline{E}} \cdot \frac{D\underline{P}}{Dt} - \hat{\underline{M}} \cdot \frac{D\underline{B}}{Dt}. \qquad (6)$$

This set of conservation laws can constitute a closed system when it is supplemented by appropriate constitutive equations for the field variables such as polarization and magnetization. The most general theory of constitutive equations determining the polarization, magnetization, electric conduction current, heat flux, and Cauchy stress tensor has been developed by Eringen and Maugin (1990a; 1990b), while the second order theory has been developed by Ko and Dulikravich (1998). If the analysis is limited to the linear fluid medium, then the following expressions can be used.

$$\underline{P} = \varepsilon_p \hat{\underline{E}}, \qquad (7)$$

$$\hat{\underline{M}} = \frac{1}{\mu_m} \underline{B}, \qquad (8)$$

$$\hat{\underline{J}} = \sigma \hat{\underline{E}} + \sigma_T \nabla T, \qquad (9)$$

$$\underline{q} = \kappa \nabla T + \kappa_E \hat{\underline{E}}, \qquad (10)$$

$$\underline{\underline{t}} = -\tilde{p} \underline{\underline{I}} + \eta [\nabla \underline{v} + (\nabla \underline{v})^t]. \qquad (11)$$

Here, coefficients $\varepsilon_p, \mu_m, \sigma, \sigma_T, \kappa, \kappa_E, \eta$ are material properties and depend only on temperature in the case of an incompressible fluid. For the physical importance of these properties, see review papers by Dulikravich and Lynn (1997a; 1997b).

Substituting the constitutive relations (Eqs. (7)-(11)) into the conservation equations (Eqs. (1)

- (6)), a closed set of governing equations can be obtained. Since the purpose of the present study is a derivation of non-reflective boundary conditions for an axisymmetric EMHD flow, a simplified version will be given in the next section.

3. Axisymmetric EMHD Flows

Here, we will deal with an axisymmetric EMHD flow case in which there is no circumferential variation of any field variable. Such flows can be most easily described by using the circular cylindrical coordinates, (r,θ,z). The velocity and electric fields (influencing $\underline{E}, \underline{\hat{E}}, \underline{P}, \underline{\hat{J}}, \underline{J}$) have only the r- and z- components, while the magnetic field (influencing $\underline{B}, \underline{H}, \underline{M}, \underline{\hat{M}}$) has only the θ- component. None of the variables depends on θ- coordinate. More specifically,

$$\underline{v}=\begin{Bmatrix}V_1\\0\\V_3\end{Bmatrix},\quad \underline{E}=\begin{Bmatrix}E_1\\0\\E_3\end{Bmatrix},\quad \underline{B}=\begin{Bmatrix}0\\B_2\\0\end{Bmatrix}. \tag{12}$$

The remaining field vectors can also be readily represented from this form of velocity, electric, and magnetic fields. For example, the polarization per unit volume and the magnetic field intensity vector are as follows.

$$\underline{P}=\begin{Bmatrix}P_1\\0\\P_3\end{Bmatrix}=\begin{Bmatrix}\varepsilon_p\hat{E}_1\\0\\\varepsilon_p\hat{E}_3\end{Bmatrix}=\begin{Bmatrix}\varepsilon_p(E_1-V_3B_2)\\0\\\varepsilon_p(E_3+V_1B_2)\end{Bmatrix} \tag{13}$$

$$\underline{H}=\begin{Bmatrix}0\\H_2\\0\end{Bmatrix}=\begin{Bmatrix}0\\\dfrac{B_2}{\mu}-\varepsilon_p(V_1E_3-V_3E_1)\\0\end{Bmatrix} \tag{14}$$

Following the theory of vector and tensor analysis in general curvilinear coordinates (Happel and Brenner, 1965) and after a step by step calculation, we can obtain following system of governing equations for an axisymmetric EMHD flow model.

$$\varepsilon_0\frac{\partial E_1}{\partial t}=-\frac{\partial H_2}{\partial z}-\sigma_T\frac{\partial T}{\partial r}-\sigma\hat{E}_1-q_eV_1-\frac{\partial P_1}{\partial t} \tag{15}$$

$$\varepsilon_0\frac{\partial E_3}{\partial t}=\frac{\partial H_2}{\partial r}+\frac{H_2}{r}-\sigma_T\frac{\partial T}{\partial z}-\sigma\hat{E}_3-q_eV_3-\frac{\partial P_3}{\partial t} \tag{16}$$

$$\frac{\partial B_2}{\partial t}=\frac{\partial E_3}{\partial r}-\frac{\partial E_1}{\partial z} \tag{17}$$

$$\begin{aligned}\frac{\partial q_e}{\partial t}=&-V_1\frac{\partial q_e}{\partial r}-V_3\frac{\partial q_e}{\partial z}-\sigma\left(\frac{\partial\hat{E}_1}{\partial r}+\frac{\hat{E}_1}{r}+\frac{\partial\hat{E}_3}{\partial z}\right)\\&-\sigma_T\left(\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2T}{\partial z^2}\right)\end{aligned} \tag{18}$$

$$\frac{\partial\tilde{p}}{\partial t}=-\beta\left(\frac{\partial V_1}{\partial r}+\frac{V_1}{r}+\frac{\partial V_3}{\partial z}\right) \tag{19}$$

$$\begin{aligned}\rho\frac{\partial V_1}{\partial t}=&-\rho\left(V_1\frac{\partial V_1}{\partial r}+V_3\frac{\partial V_1}{\partial z}\right)-\frac{\partial\tilde{p}}{\partial r}+\rho f_1\\&+\eta\left(\frac{\partial^2V_1}{\partial r^2}+\frac{1}{r}\frac{\partial V_1}{\partial r}+\frac{\partial^2V_1}{\partial z^2}-\frac{V_1}{r^2}\right)+q_e\hat{E}_1-\sigma\hat{E}_3B_2\\&-\sigma_TB_2\frac{\partial T}{\partial z}+P_1\frac{\partial E_1}{\partial r}+P_3\frac{\partial E_1}{\partial z}-V_1\frac{\partial(P_3B_2)}{\partial r}\\&-V_3\frac{\partial(P_3B_2)}{\partial z}+\left(\frac{B_2}{\mu_m}+\varepsilon_p(V_1E_3-V_3E_1)\right)\frac{\partial B_2}{\partial r}\\&-B_2\frac{\partial P_3}{\partial t}\end{aligned} \tag{20}$$

$$\begin{aligned}\rho\frac{\partial V_3}{\partial t}=&-\rho\left(V_1\frac{\partial V_3}{\partial r}+V_3\frac{\partial V_3}{\partial z}\right)-\frac{\partial\tilde{p}}{\partial z}+\rho f_3\\&+\eta\left(\frac{\partial^2V_3}{\partial r^2}+\frac{1}{r}\frac{\partial V_3}{\partial r}+\frac{\partial^2V_3}{\partial z^2}\right)+q_e\hat{E}_3+\sigma\hat{E}_1B_2\\&+\sigma_TB_2\frac{\partial T}{\partial r}+P_1\frac{\partial E_3}{\partial r}+P_3\frac{\partial E_3}{\partial z}+V_1\frac{\partial(P_1B_2)}{\partial r}\\&+V_3\frac{\partial(P_1B_2)}{\partial z}+\left(\frac{B_2}{\mu_m}+\varepsilon_p(V_1E_3-V_3E_1)\right)\frac{\partial B_2}{\partial z}\\&+B_2\frac{\partial P_1}{\partial t}\end{aligned} \tag{21}$$

$$\begin{aligned}\rho c_p\frac{\partial T}{\partial t}=&-\rho c_p\left(V_1\frac{\partial T}{\partial r}+V_3\frac{\partial T}{\partial z}\right)+Q_h+\kappa\left(\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2T}{\partial z^2}\right)\\&+\kappa_E\left(\frac{\partial\hat{E}_1}{\partial r}+\frac{\hat{E}_1}{r}+\frac{\partial\hat{E}_1}{\partial z}\right)+\sigma_T\left(\hat{E}_1\frac{\partial T}{\partial r}+\hat{E}_3\frac{\partial T}{\partial z}\right)\\&+\sigma(\hat{E}_1^2+\hat{E}_3^2)+\hat{E}_1\left(V_1\frac{\partial P_1}{\partial r}+V_3\frac{\partial P_1}{\partial z}\right)\\&+\hat{E}_3\left(V_1\frac{\partial P_3}{\partial r}+V_3\frac{\partial P_3}{\partial z}\right)+\hat{E}_1\frac{\partial P_1}{\partial t}+\hat{E}_3\frac{\partial P_3}{\partial t}\\&-\frac{B_2}{\mu_m}\left(\frac{\partial E_3}{\partial r}-\frac{\partial E_1}{\partial z}+V_1\frac{\partial B_2}{\partial r}+V_3\frac{\partial B_2}{\partial z}\right)\end{aligned} \tag{22}$$

Here, the variation of material properties such as electric permittivity, magnetic permeability, and viscosity are assumed negligible. Notice also that an artificial compressibility term is added in the continuity equation in order to create an artificial unsteady term for marching in time (Chorin, 1967). This modification is justified if the aim is to get a steady-state solution.

The right hand sides of Eqs. (15)-(16) and (20)-(22) contain unsteady terms $\partial P_1/\partial t$ and $\partial P_3/\partial t$. Even if it is assumed that the polarization electric permittivity ε_p is constant, these terms must be solved for using the definition given in Eq. (13). Solving Eqs. (15)-(21) with the defining Eq. (13) simultaneously for all the rate of time variation terms is possible in principle. The resulting system, however, is extremely complicated and severely coupled. It is nearly impossible to elucidate the physical meaning of the characteristic behavior of such EMHD system, because the system does not allow further analytical treatment.

If the analysis is limited to the formulation of non-reflective boundary conditions needed in simulating the steady state flows only, then this difficulty can be avoided by slightly modifying the governing equations. The time derivative of the polarization vector on the right hand sides of Eqs. (15)-(16) and (20)-(22) can be omitted in case of simulation of steady-state EMHD flows, because these terms vanish at the converged final steady state. This idea is similar to the preconditioning method that is frequently used to accelerate the convergence of iterative algorithms as well as the artificial compressibility method. It has been successfully adopted in formulating non-reflective boundary conditions for a two-dimensional planar EMHD flow model (Ko and Dulikravich, 1999).

The modified version of the evolution equations (15)-(22), which can be readily obtained by omitting the unsteady terms $\partial P_1/\partial t$ and $\partial P_3/\partial t$, will be used to investigate the characteristic features of the axisymmetric EMHD flows and to obtain non-reflective boundary conditions at the open boundaries for such flows.

4. Non-Reflective Boundary Conditions

In order to obtain a complete understanding of the characteristic behavior of an axisymmetric EMHD flow, it is desirable to investigate the full EMHD evolution equations. But the coefficient matrix of the characteristic form of the full governing equation system is so dense that it does not allow a symbolic (closed form) calculation of eigenvalues. In addition, electro-magnetic time scale differs from the thermo-mechanical one by orders of magnitude. Except for the extreme cases such as a flow in a medium without polarizaion and magnetization, it does not seem possible to derive non-reflective boundary conditions by treating the electro-magnetic and thermo-mechanical effects in a coupled manner. Therefore, the formulation of the characteristic boundary conditions will be carried out separately for the two subsystems (electro-magnetic and thermo-mechanical). This approach is acceptable when computing steady-state solution of EMHD flows.

Non-reflective Boundary Conditions for Electro-magnetic Subsystem

As mentioned in the previous section, the time derivatives of polarization will be omitted here and in the next subsection. A characteristic form of the modified electro-magnetic (Maxwell's) subsystem can be written from Eqs. (15)-(18) as follows.

$$\frac{\partial \underline{Q}^1}{\partial t} + \underline{\underline{A}}^{11}\frac{\partial \underline{Q}^1}{\partial r} + \underline{\underline{B}}^{11}\frac{\partial \underline{Q}^1}{\partial z} = \underline{S}^1 - \underline{\underline{A}}^{12}\frac{\partial \underline{Q}^2}{\partial r} - \underline{\underline{B}}^{12}\frac{\partial \underline{Q}^2}{\partial z} \equiv \underline{\tilde{S}}^1 \quad (23)$$

Here, indices 1 and 2 correspond to the electro-magnetic subsystem and thermo-mechanical subsystem, respectively. The electro-magnetic solution vector $\underline{Q}^1$ contains the following unknown primitive variables

$$\underline{Q}^1 = \{E_1, E_3, B_2, q_e\}^t \quad (24)$$

while $\underline{Q}^2$ denotes the solution vector representing thermo-mechanical field. The coefficient matrices $\underline{\underline{A}}^{11}$ and $\underline{\underline{B}}^{11}$ are defined by

$$\underline{\underline{A}}^{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -(\varepsilon_r - 1)V_3 & (\varepsilon_r - 1)V_1 & -1/\varepsilon_0\mu & 0 \\ 0 & -1 & 0 & 0 \\ \sigma & 0 & -\sigma V_3 & V_1 \end{bmatrix}, \quad (25)$$

$$\underline{\underline{B}}^{11} = \begin{bmatrix} (\varepsilon_r - 1)V_3 & -(\varepsilon_r - 1)V_1 & 1/\varepsilon_0\mu & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sigma & \sigma V_1 & V_3 \end{bmatrix}, \quad (26)$$

The remaining coefficient matrices, $\underline{\underline{A}}^{12}$ and $\underline{\underline{B}}^{12}$, may be obtained by collecting the appropriate first order spatial derivative terms in Eqs. (15)-(18). The source vector $\underline{S}^1$ is defined as

$$\underline{S}^1 = \begin{Bmatrix} -(q_e V_1 + \sigma \hat{E}_1)/\varepsilon_0 \\ (H_2/r - q_e V_3 - \sigma \hat{E}_3)/\varepsilon_0 \\ 0 \\ -\sigma_T \tilde{\nabla}^2 T - \sigma \hat{E}_1/r \end{Bmatrix} \quad (27)$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \quad (28)$$

is a second order differential operator which is similar to the Laplacian operator. In a strict sense, neither $\underline{S}^1$ nor $\tilde{\underline{S}}^1$ (see Eq. (23)) can be a source vector because a source vector should not contain any spatial derivative. A second order derivative term associated with the diffusion of heat is included in these two source vectors. This means that the diffusive nature of the characteristic waves has been neglected. This is the only viable option, since there is no known method to handle this effect.

The eigenmatrix $\underline{\underline{\Lambda}}(\underline{\underline{A}}^{11})$ corresponding to the coefficient matrix $\underline{\underline{A}}^{11}$ can be calculated as

$$\underline{\underline{\Lambda}}(\underline{\underline{A}}^{11}) = \mathrm{diag}[\lambda_1^{A^{11}}, \lambda_2^{A^{11}}, \lambda_3^{A^{11}}, \lambda_4^{A^{11}}] = \mathrm{diag}[0, V_1, \xi_1^A, \xi_2^A] \quad (29)$$

$$\xi_{1,2}^A = \frac{\sqrt{\mu}\varepsilon_p V_1 \pm \sqrt{4\varepsilon_0 + \mu\varepsilon_p^2 V_1^2}}{2\varepsilon_0\sqrt{\mu}}, \quad (30)$$

while the eigenmatrix corresponding to $\underline{\underline{B}}^{11}$ is found to be

$$\underline{\underline{\Lambda}}(\underline{\underline{B}}^{11}) = \mathrm{diag}\{\lambda_1^{B^{11}}, \lambda_2^{B^{11}}, \lambda_3^{B^{11}}, \lambda_4^{B^{11}}\} = \mathrm{diag}\{0, V_3, \xi_1^B, \xi_2^B\} \quad (31)$$

$$\xi_{1,2}^B = \frac{\sqrt{\mu}\varepsilon_p V_3 \pm \sqrt{4\varepsilon_0 + \mu\varepsilon_p^2 V_3^2}}{2\varepsilon_0\sqrt{\mu}}. \quad (32)$$

As expected, all of the eigenvalues are real, which means that the electro-magnetic evolution equation system is hyperbolic. The characteristic wave propagating in the radial direction (represented by $\underline{\underline{A}}^{11}$) shows the same behavior as the axially propagating characteristic wave (represented by $\underline{\underline{B}}^{11}$). The only difference between Eq. (29) and Eq. (31) is that V_1 was replaced by V_3. These eigenvalues show that the incoming and outgoing electro-magnetic waves are not influenced by the electric or magnetic field, but by the fluid velocity, magnetic permeability, and electric permittivity of the fluid.

Since the wave propagation characteristics in the axial z- direction are the same as those in the radial r- direction, it is sufficient to set up the non-reflective boundary conditions on the surfaces r = constant, that is, in the radial direction. The matrix of left eigenvectors corresponding to $\underline{\underline{A}}^{11}$, which is identical to the inverse of the similarity transformation matrix, can be calculated as

$$\underline{\underline{N}}^{A^{11}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\sigma(a_1 + \mu\varepsilon_p V_3^2)}{a_1 V_1} & -\frac{\sigma\mu\varepsilon_0 V_3}{a_1} & \frac{\sigma\mu(\varepsilon_p - \varepsilon_0)V_1 V_3}{a_1} & 1 \\ \mu\varepsilon_p V_3 & -\mu\varepsilon_0\xi_1^A & 1 & 0 \\ \mu\varepsilon_p V_3 & -\mu\varepsilon_0\xi_2^A & 1 & 0 \end{bmatrix} \quad (33)$$

$$a_1 = 1 + (\varepsilon_p - \varepsilon_0)\mu V_1^2. \quad (34)$$

The direction of wave propagation is well defined for the actually or locally one-dimensional problems. For two-dimensional problems, there is no unique direction of propagation. This is reflected in the fact that $\underline{\underline{A}}^{11}$ and $\underline{\underline{B}}^{11}$ cannot be simultaneously diagonalized.

For most cases, however, there exists a main flow direction at the open (inlet and outlet) boundaries where the characteristic boundary conditions are required. If r- direction is such a direction, then one can estimate that the wave propagation characteristics are predominantly determined by $\underline{\underline{A}}^{11}$, because the effect of the transversal variation is relatively negligible. In this (quasi one-dimensional) approach, the transverse term is considered constant and treated as a source term in the same way as the thermo-mechanical derivative terms are treated as source terms (Thompson, 1987; Dulikravich and Jing, 1997).

The fact that all eigenvalues of $\underline{\underline{A}}^{11}$ are real means that Eq. (23) is locally hyperbolic. Following Thompson's approach, the boundary conditions at the inlet and outlet can be written as

$$\underline{n}_i^{A^{11}} \frac{\partial \underline{Q}^1}{\partial t} + \delta_i \lambda_i^{A^{11}} \underline{n}_i^{A^{11}} \frac{\partial \underline{Q}^1}{\partial r} + \underline{n}_i^{A^{11}} \underline{U}^1 = 0 \quad \text{at} \quad r = r_{in},\ r_{out}\,. \tag{35}$$

Here, $\underline{n}_i^{A^{11}}$ is the i-th row vector of $\underline{\underline{N}}^{A^{11}}$, and δ_i is defined by

$$\delta_i = \begin{cases} 1 \text{ for outgoing waves,} \\ 0 \text{ for incoming waves.} \end{cases} \tag{36}$$

Finally, $\underline{U}^1$ is the source term modification vector, which is given by

$$\underline{U}^1 = \underline{\underline{B}}^{11} \frac{\partial \underline{Q}^1}{\partial z} - \underline{\tilde{S}}^1\,. \tag{37}$$

In cases where the open boundaries lie at the axial cross-sections, that is, defined by z = constant, the non-reflective boundary conditions can be written as

$$\underline{n}_i^{B^{11}} \frac{\partial \underline{Q}^1}{\partial t} + \delta_i \lambda_i^{B^{11}} \underline{n}_i^{B^{11}} \frac{\partial \underline{Q}^1}{\partial z} + \underline{n}_i^{B^{11}} \underline{W}^1 = 0 \quad \text{at} \quad z = z_{in},\ z_{out}\,. \tag{38}$$

Here, $\underline{n}_i^{B^{11}}$ is the i-th row vector of $\underline{\underline{N}}^{B^{11}}$ defined by

$$\underline{\underline{N}}^{B^{11}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{\sigma\mu\varepsilon_0 V_1}{b_1} & \dfrac{\sigma(b_1 + \mu\varepsilon_p V_1^2)}{b_1 V_3} & \dfrac{\sigma\mu(\varepsilon_p - \varepsilon_0) V_1 V_3}{b_1} & 1 \\ \mu\varepsilon_0 \xi_1^B & -\mu\varepsilon_p V_1 & 1 & 0 \\ \mu\varepsilon_0 \xi_2^B & -\mu\varepsilon_p V & 1 & 0 \end{bmatrix} \tag{39}$$

$$b_1 = 1 + (\varepsilon_p - \varepsilon_0)\mu V_3^2\,. \tag{40}$$

The vector, $\underline{W}^1$, which modifies the source term is given by

$$\underline{W}^1 = \underline{\underline{A}}^{11} \frac{\partial \underline{Q}^1}{\partial r} - \underline{\tilde{S}}^1\,. \tag{41}$$

Note the close similarity, which exists between the radial mode and the axial mode, represented by Eqs. (25) through (41). The curvature effect of the boundary surface, that is the dependence on r, appears only in a source term ($\underline{S}^1$ or $\underline{\tilde{S}}^1$).

Non-reflective Boundary Conditions for Thermo-mechanical Subsystem

The procedure of formulating the non-reflective boundary conditions for the thermo-mechanical subsystem is the same as that for the electro-magnetic subsystem. Using Eqs. (19)-(22), this subsystem can be written in a characteristic form as

$$\frac{\partial \underline{Q}^2}{\partial t} + \underline{\underline{A}}^{22} \frac{\partial \underline{Q}^2}{\partial r} + \underline{\underline{B}}^{22} \frac{\partial \underline{Q}^2}{\partial z} = \underline{S}^2 - \underline{\underline{A}}^{21} \frac{\partial \underline{Q}^1}{\partial r} - \underline{\underline{B}}^{21} \frac{\partial \underline{Q}^1}{\partial z} \equiv \underline{\tilde{S}}^2 \tag{42}$$

where the solution vector $\underline{Q}^2$ and source vector $\underline{S}^2$ are defined as

$$\underline{Q}^2 = \{\tilde{p},\ V_1,\ V_3,\ T\}^t, \tag{43}$$

$$\underline{S}^2 = \begin{Bmatrix} -\beta V_1 / r \\ \dfrac{\eta(\tilde{\nabla}^2 V_1 - V_1/r^2) + q_e \hat{E}_1 - \sigma \hat{E}_3 B_2 + \rho f_1}{\rho} \\ \dfrac{\eta \tilde{\nabla}^2 V_3 + q_e \hat{E}_3 + \sigma \hat{E}_1 B_2 + \rho f_3}{\rho} \\ \dfrac{Q_h + \kappa \tilde{\nabla}^2 T + \kappa_E \hat{E}_1 / r + \sigma(\hat{E}_1^2 + \hat{E}_3^2)}{\rho c_p} \end{Bmatrix}, \tag{44}$$

respectively. As in the electro-magnetic subsystem, the source vector has second order derivative terms due to the momentum diffusion and heat diffusion. We will assume that their effects are much smaller than the first order derivative terms. The coefficient matrices $\underline{\underline{A}}^{22}$, $\underline{\underline{B}}^{22}$ are defined as follows:

$$\underline{\underline{A}}^{22} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ \frac{1}{\rho} & GV_1 - \frac{\eta}{\rho r} & 0 & 0 \\ 0 & 0 & GV_1 - \frac{\eta}{\rho r} & -\frac{\sigma_T B_2}{\rho} \\ 0 & -\frac{P_3 V_1 B_2}{\rho c_p} & \frac{(P_1 V_1 + \kappa_E) B_2}{\rho c_p} & V_1 - \frac{\sigma_T \hat{E}_1 + \frac{\kappa}{r}}{\rho c_p} \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1/\rho & c_1 & 0 & 0 \\ 0 & 0 & c_1 & -c_2 \\ 0 & -c_3 & c_4 & c_5 \end{bmatrix} \tag{45}$$

$$\underline{\underline{B}}^{22} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & GV_3 & 0 & \frac{\sigma_T B_2}{\rho} \\ \frac{1}{\rho} & 0 & GV_3 & 0 \\ 0 & -\frac{(P_3 V_3 + \kappa_E) B_2}{\rho c_p} & \frac{P_1 V_3 B_2}{\rho c_p} & V_3 - \frac{\sigma_T \hat{E}_3}{\rho c_p} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & d_1 & 0 & d_2 \\ 1/\rho & 0 & d_1 & 0 \\ 0 & -d_4 & d_3 & d_5 \end{bmatrix} \tag{46}$$

$$G = 1 + \frac{\varepsilon_p B_2^2}{\rho}. \tag{47}$$

It should be noted that the most important factor combing the electro-magnetic and thermo-mechanical effects is the thermo-electric conductivity, σ_T.

In contrast to the electro-magnetic subsystem, the coefficient matrix $\underline{\underline{B}}^{22}$ associated with the axial direction cannot be constructed cyclically from $\underline{\underline{A}}^{22}$ associated with the radial direction. The matrix $\underline{\underline{A}}^{22}$ has terms proportional to $1/r$ in its principal diagonal elements (c_1 and c_5). These terms arise from the (momentum and heat) diffusion and reflect the curvature effect of the cylindrical boundary surfaces. In a far field this effect diminishes to zero and the situation arising in a two-dimensional planar case is recovered. In a near filed, however, the effect of curved surface is important.

The eigenmatrix $\underline{\underline{\Lambda}}(\underline{\underline{A}}^{22})$ corresponding to $\underline{\underline{A}}^{22}$ can be written as

$$\underline{\underline{\Lambda}}(\underline{\underline{A}}^{22}) = \mathrm{diag}[\lambda_1^{A^{22}}, \lambda_2^{A^{22}}, \lambda_3^{A^{22}}, \lambda_4^{A^{22}}] = \mathrm{diag}[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \tag{48}$$

$$\lambda_{1,2}^{A^{22}} = (c_1 \pm c_6)/2, \quad \lambda_{3,4}^{A^{22}} = (c_1 + c_5 \pm c_7)/2, \tag{49}$$

$$c_6 = \sqrt{4\beta/\rho + c_1^2}, \quad c_7 = \sqrt{(c_1 - c_5)^2 - 4c_2 c_4}. \tag{50}$$

The eigenmatrix $\underline{\underline{\Lambda}}(\underline{\underline{B}}^{22})$ corresponding to $\underline{\underline{B}}^{22}$ can be written as

$$\underline{\underline{\Lambda}}(\underline{\underline{B}}^{22}) = \mathrm{diag}[\lambda_1^{B^{22}}, \lambda_2^{B^{22}}, \lambda_3^{B^{22}}, \lambda_4^{B^{22}}] = \mathrm{diag}[\beta_1, \beta_2, \beta_3, \beta_4] \tag{51}$$

$$\lambda_{1,2}^{B^{22}} = (d_1 \pm d_6)/2, \quad \lambda_{3,4}^{B^{22}} = (d_1 + d_5 \pm d_7)/2, \tag{52}$$

$$d_6 = \sqrt{4\beta/\rho + d_1^2}, \quad d_7 = \sqrt{(d_1 - d_5)^2 - 4d_2 d_4}. \tag{53}$$

From the results of eigenvalues, the wave speed propagating in the radial direction is a function of coordinate, r, at least in a near field. On the other hand, the wave speed in the axial direction is not influenced by radial distance itself. In the far field, that is, as $r \to \infty$, the qualitative difference between two modes diminishes. Their variation with respect to position is also included indirectly in the field strengths. The eigenvalues for both r- and z- directions are strongly dependent on the electric and magnetic field intensities, velocity, and fluid material property such as electric conductivity, density, and magnetic permittivity. This behavior is quite different from the electro-magnetic subsystem, and is evidence that the effect of electro-magnetic field on flow is much stronger than that of flow-field on the electro-magnetic field.

The non-reflective boundary conditions for the thermo-mechanical subsystem can be derived in exactly the same way as for the electro-magnetic subsystem. They can be expressed concisely as

$$\underline{n}_i^{A^{22}} \frac{\partial \underline{Q}^2}{\partial t} + \delta_i \lambda_i^{A^{22}} \underline{n}_i^{A^{22}} \frac{\partial \underline{Q}^2}{\partial r} + \underline{n}_i^{A^{22}} \underline{U}^2 = 0 \quad \text{at} \quad r = r_{in},\ r_{out}, \tag{54}$$

$$\underline{n}_i^{B^{22}} \frac{\partial \underline{Q}^2}{\partial t} + \delta_i \lambda_i^{B^{22}} \underline{n}_i^{B^{22}} \frac{\partial \underline{Q}^2}{\partial z} + \underline{n}_i^{B^{22}} \underline{W}^2 = 0 \ \text{at} \quad z = z_{in},\ z_{out}. \tag{55}$$

In this expression, $\underline{n}_i^{A^{22}}$ and $\underline{n}_i^{B^{22}}$ denote the i-th row vector of the similarity transformation matrices $\underline{\underline{N}}^{A^{22}}$ and $\underline{\underline{N}}^{B^{22}}$ defined by

$$\underline{\underline{N}}^{A^{22}} = \begin{bmatrix} -\alpha_2/\beta & 1 & 0 & 0 \\ -\alpha_1/\beta & 1 & 0 & 0 \\ \dfrac{2c_3}{2\beta + (c_8 - c_5 c_7)\rho} & \dfrac{-2\rho c_3 \alpha_3}{2\beta + (c_8 - c_5 c_7)\rho} & \dfrac{c_5 - c_1 - c_7}{2c_2} & 1 \\ \dfrac{2c_3}{2\beta + (c_8 + c_5 c_7)\rho} & \dfrac{2\rho c_3 \alpha_4}{2\beta + (c_8 - c_5 c_7)\rho} & \dfrac{c_5 - c_1 + c_7}{2c_2} & 1 \end{bmatrix} \tag{56}$$

$$c_8 = 2c_2 c_4 + c_1 c_5 - c_5^2 \tag{57}$$

and

$$\underline{\underline{N}}^{B^{22}} = \begin{bmatrix} -\beta_2/\beta & 0 & 1 & 0 \\ -\beta_1/\beta & 0 & 1 & 0 \\ \dfrac{-2d_3}{2\beta + (d_8 - d_5 d_7)\rho} & \dfrac{d_1 - d_5 + d_7}{2d_2} & \dfrac{-2\rho d_3 \alpha_3}{2\beta + (d_8 - d_5 d_7)\rho} & 1 \\ \dfrac{-2d_3}{2\beta + (d_8 + d_5 d_7)\rho} & \dfrac{d_1 - d_5 - d_7}{2d_2} & \dfrac{-2\rho d_3 \alpha_4}{2\beta + (d_8 + d_5 d_7)\rho} & 1 \end{bmatrix} \tag{58}$$

$$d_8 = 2d_2 d_4 + d_1 d_5 - d_5^2, \tag{59}$$

respectively. In addition, source terms should be modified as

$$\underline{U}^2 = \underline{\underline{B}}^{22} \frac{\partial \underline{Q}^2}{\partial z} - \underline{\tilde{S}}^2, \tag{60}$$

$$\underline{W}^2 = \underline{\underline{A}}^{22} \frac{\partial \underline{Q}^2}{\partial r} - \underline{\tilde{S}}^2. \tag{61}$$

Conclusion

Starting from a consistent linear EMHD model written in a circular cylindrical coordinate system, the derivation of non-reflective boundary conditions for numerical simulation of steady axisymmetric EMHD flows was performed. An artificial compressibility term was added in the continuity equation, and the temporal variation of polarization vector was neglected. Under these simplifications, both modified Maxwell equations (electro-magnetic subsystem) and modified Navier-Stokes equations (thermo-mechanical subsystem) were transformed to their characteristic form for the fluids with constant material property. The characteristic and non-reflective boundary conditions at the inlet and outlet boundary were formulated for both subsystems by assuming that transversal variation is negligible.

The results for the eigenvalues of the coefficient matrices show that there is a substantial coupling between the electro-magnetic field and flow-field. The effect of curvature of the radial boundary surfaces is manifested in the radially propagating waves. This effect reduces to zero as the far field is reached. In the far field, there is no distinction between the waves in the radial direction and the axial direction. Because of the modification of some unsteady terms, our results are only valid for steady-state problems. Therefore, further studies are needed to circumvent this restriction for the simulation of unsteady flows.

Acknowledgments

The authors are grateful for the partial support by the ALCOA Foundation Grant facilitated by Dr. Yimin Ruan and Dr. Owen Richmond of ALCOA Technical Center, and a Lawrence Livermore National Laboratory grant LLNL-B344847 monitored by Dr. J. Ray Smith.

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