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A fully non-linear theory of electro-magneto-hydrodynamics

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Abstract

A number of analytical models exist for both electrohydrodynamics (EHD) and magnetohydrodynamics (MHD). At present there are no practical yet consistent models for the combined electro-magneto-hydrodynamic (EMHD) effects which occur quite often in actual situations. This work represents an attempt to develop such a fully consistent analytical model for multi-dimensional, steady and unsteady, compressible and incompressible flows of electrically conducting fluids under the simultaneous or separate influence of externally applied and internally generated electric and magnetic fields. The approach is based on the fundamental laws of continuum mechanics and thermodynamics with all assumptions clearly stated and consistently applied. The resulting second order EMHD model allows for non-linear electric and thermal conduction and electromagnetic stress within the medium. The new model is therefore superior to the existing EMHD models and represents a tractable set of equations suitable for detailed numerical discretization and integration. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Electrohydrodynamics; Magnetohydrodynamics; Consistent non-linear model

1. Introduction

The study of fluid flows under the influence of the externally applied and internally generated electric and magnetic fields is often called electromagneto dynamics of fluids [1], electromagneto fluid dynamics [2,3], electromagneto hydrodynamics (EMHD) [4–6], magneto-gas dynamics and plasma dynamics [7], or the electrodynamics of material continua [8]. To reduce the complexity of the model for this phenomenon, the analytical models have traditionally been simplified into electrohydrodynamics (EHD) and magnetohydrodynamics (MHD), i.e. the study of flows containing electric charges under the influence of an externally applied electric field and negligible magnetic field and the study of flows influenced only by an externally applied magnetic field without electric charges in the fluid [9–11]. The existing EHD and MHD models often represent unacceptable oversimplifications of the actual combined electromagnetic effects [3]. More recently, rigorous continuum mechanical treatments of unified electromagneto gas dynamic (EMGD) [12] and EMHD flows [3–6,13,14] have been developed. These approaches are limited to non-relativistic, quasi-static, or relatively low-frequency phenomenon [15,16]. The existing EMGD model is extremely complex and requires a large number of physical properties of the fluid, most of which are still

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Nomenclature

$\mathbf{B}$	$(=\mu_0(\mathbf{H}+\mathbf{M}))$, magnetic flux density, kg $A^{-1}s^{-2}$
B_0	reference value of $\mathbf{B}$, kg A^{-1} s^{-2}
c	$(=3\times10^8)$, speed of light in vacuum, m s^{-1}
C_p	specific heat at constant pressure, K^{-1} m^2 s^{-2}
$\mathbf{d}$	$\{=\frac{1}{2}[(\nabla\mathbf{v}+(\nabla\mathbf{v})^t]\}$, average rate of deformation tensor, s^{-1}
$\frac{D}{Dt}$	$\left(=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)$, material derivative, s^{-1}
$\mathbf{D}$	$(=\varepsilon_0\mathbf{E}+\mathbf{P})$, electric displacement vector, A s m^{-2}
$\mathbf{E}$	electric field intensity, kg m s^{-3} A^{-1}
E_0	reference value of $\mathbf{E}$, kg m s^{-3} A^{-1}
$\hat{\mathbf{E}}$	$(=\mathbf{E}+\mathbf{v}\times\mathbf{B})$, electromotive intensity, kg m s^{-3} A^{-1}
$\mathbf{f}$	mechanical body force per unit mass, m s^{-2}
$\mathbf{F}^{em}$	electromagnetic body force per unit volume, kg m^{-2} s^{-2}
g	acceleration due to gravity, m s^{-2}
$\mathbf{H}$	magnetic field intensity, A m^{-1}
$\mathbf{I}$	unit tensor
$\mathbf{J}$	$(=\mathbf{J}_c+q_e\mathbf{v})$, electric current density, A m^{-2}
$\mathbf{J}_c$	electric conduction current, A m^{-2}
L	reference length, m
$\mathbf{M}$	total magnetization per unit volume, A m^{-1}
$\hat{\mathbf{M}}$	$(=\mathbf{M}+\mathbf{v}\times\mathbf{P})$, magnetomotive intensity, A m^{-1}
p	pressure, kg m^{-1} s^{-2}
$\mathbf{P}$	total polarization per unit volume, A s m^{-2}
q_e	total electric charge per unit volume, A s m^{-3}
$\mathbf{q}$	conduction heat flux, kg s^{-3}
Q_h	heat source per unit volume, kg m^{-1} s^{-3}
s	entropy per unit mass, m^2 kg^{-1} K^{-1} s^{-2}
t	time, s
$\mathbf{t}$	$(=\varphi\mathbf{I}+\boldsymbol{\tau})$, Cauchy stress tensor, kg m^{-1} s^{-2}
T	absolute temperature, K
T_0	reference temperature, K
ΔT	reference temperature difference, K
$\hat{u}$	internal energy per unit mass, m^2 s^{-2}
$\mathbf{v}$	fluid velocity, m s^{-1}
V	reference speed, m s^{-1}

Greek symbols

β	$(=\varepsilon_0\mu_0V^2)$, square of electro-magnetic Mach number
ε	electric permittivity, kg^{-1} m^{-3} s^4 A^2
ε_0	$(=8.854\times10^{-12})$, electric permittivity of vacuum, kg^{-1} m^{-3} s^4 A^2
ε_p	$(=\varepsilon_0\chi_e)$, polarization electric permittivity, kg^{-1} m^{-3} s^4 A^2
ε_r	$(=\varepsilon/\varepsilon_0)$, relative electric permittivity
ε_{klm}	permutation symbol
φ	modified hydrostatic pressure, kg m^{-1} s^{-2}

ρ	fluid density, kg m^{-3}
τ	deviator part of stress tensor, kg m^{-1} s^{-2}
θ	dimensionless temperature
μ	magnetic permeability, kg m A^{-2} s^{-2}
μ_0	$(=4\pi\times 10^{-7})$, magnetic permeability of vacuum, kg m A^{-2} s^{-2}
μ_r	$(=\mu/\mu_0)$, relative magnetic permeability
μ_m	$(=\mu_0/\chi_B)$, magnetization magnetic permeability, kg m A^{-2} s^{-2}
χ_B	$(=1-\mu_r^{-1})$, magnetic susceptibility based on B
χ_e	$(=\varepsilon_r-1)$, electric susceptibility
Ψ	$(=\hat{u}-Ts-\hat{\mathbf{E}}\cdot\mathbf{P}/\rho)$, generalized Helmholtz free energy per unit mass, m^2 s^{-2}

unknown. Even in the case of EMHD (incompressible fluids under the influence of combined electric and magnetic fields) the existing models are not simple and are not fully consistent with the general EMGD model. Dulikravich and Jing [5,6] have shown that a compact vector form of the unified EMGD system can be written as a combination of Maxwell's electromagnetic subsystem and the Navier–Stokes fluid flow subsystem. Nevertheless, their version of the EMGD and especially of EMHD is not fully consistent with the most general version obtained by Eringen and Maugin [12]. In addition, the inconsistent models allow only for linear polarization and linear magnetization of the fluid.

The objective of this paper is to summarize the most general EMGD analytical model, develop a rational second-order approximation to the EMGD model, and finally to develop a fully consistent EMHD model. The new model should supercede the existing inconsistent EMHD model [6] and be acceptable for numerical discretization and integration. The consistent simplification of the EMHD model will also be performed using non-dimensionalization of each term in the governing equations.

2. General balance laws

The firm foundations of the EMGD theory were formulated by Eringen and Maugin [12], based on continuum mechanics. The general set of balance laws governing single-phase EMGD and EMHD flows consist of two groups of equations. One group is for the electromagnetic field, the other is for the thermomechanical field. This set should be supplemented by the material constitutive equations.

The electromagnetic balance laws are represented by the Maxwell's equations. Using a vector operator form, the Maxwell's equations can be written in the rationalized MKS system as follows:

$$\nabla\cdot\mathbf{D}=q_e \quad \text{(Gauss' law)}, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0 \quad \text{(conservation of magnetic flux)}, \tag{2}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad \text{(Faraday's law)}, \tag{3}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J} \quad \text{(Ampere–Maxwell's law)}. \tag{4}$$

This set of equations defines the divergence and curl of the electric and magnetic field. It is well known that a vector field can be completely determined if its divergence and curl are known. The relations between flux density and field intensity vectors in the polarizable and magnetizable medium are given by

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}, \tag{5}$$

$$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}). \tag{6}$$

Polarization and magnetization strongly depend on the material and can be estimated from the constitutive equations. The electric charge conservation equation is derived from a combination of

Gauss' law and Ampere–Maxwell's law as

$$\frac{\partial q_c}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{7}$$

This equation accounts for all types of charged species together since charge transport takes place by charge carrier motion and by charge jumping from one carrier to another.

The balance laws of the thermomechanical field are comprised of three conservation laws and the second law of thermodynamics. The equations of conservation of mass, linear momentum, and energy are expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{8}$$

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \mathbf{t} + \rho \mathbf{f} + \mathbf{F}^{\mathrm{em}}, \tag{9}$$

$$\rho \frac{\mathrm{D}\hat{u}}{\mathrm{D}t} = \mathbf{t} : \mathbf{d} + Q_{\mathrm{h}} + \nabla \cdot \mathbf{q} + \rho \hat{\mathbf{E}} \cdot \frac{\mathrm{D}(\rho^{-1}\mathbf{P})}{\mathrm{D}t} - \hat{\mathbf{M}} \cdot \frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t} + \mathbf{J}_c \cdot \hat{\mathbf{E}}, \tag{10}$$

respectively. Here, the electromagnetic body force per unit volume is given by

$$\mathbf{F}^{\mathrm{em}} = q_e \mathbf{E} + \mathbf{J} \times \mathbf{B} + (\nabla \mathbf{E}) \cdot \mathbf{P} + (\nabla \mathbf{B}) \cdot \mathbf{M} + \nabla[\mathbf{v}(\mathbf{P} \times \mathbf{B})] + \frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}). \tag{11}$$

The final balance law comes from the second law of thermodynamics. This is represented by the Clausius–Duhem (C–D) inequality

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} - \nabla \cdot (T^{-1}\mathbf{q}) - T^{-1} Q_{\mathrm{h}} \geqslant 0. \tag{12}$$

By using the energy conservation (Eq. (10)), and by introducing the generalized Helmholtz free energy,

$$\Psi = \hat{u} - Ts - \rho^{-1} \hat{\mathbf{E}} \cdot \mathbf{P}, \tag{13}$$

the C–D inequality can be rewritten as

$$\rho \gamma \equiv -\rho \left(\frac{\mathrm{D}\Psi}{\mathrm{D}t} + s \frac{\mathrm{D}T}{\mathrm{D}t} \right) + \mathbf{t} : \mathbf{d} + T^{-1} \mathbf{q} \cdot \nabla T - \mathbf{P} \cdot \frac{\mathrm{D}\hat{\mathbf{E}}}{\mathrm{D}t} - \hat{\mathbf{M}} \cdot \frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t} + \mathbf{J}_c \cdot \hat{\mathbf{E}} \geqslant 0. \tag{14}$$

3. Constitutive equations

3.1. General constitutive relations

In order to completely determine the electromagnetic and thermo-mechanical fields, the balance laws must be supplemented by the constitutive equations, since the number of unknowns is larger than the number of balance equations. Given the mechanical body force and the internal heat source, additional information about the polarization and magnetization, stress tensor, electric conduction, and conduction heat transfer are required. The most general theory of constitutive equations, relating the electromagnetic and thermomechanical effects, has been developed by Eringen and Maugin [12] using a continuum approach. Since the second-order model starts with the general non-linear theory, some of the essentials will be reproduced here.

For simple rate-dependent, memory-independent, and isotropic fluids, the set of independent variables for the constitutive relations can be chosen to be $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T$ and T, ρ. In other words, any field variable (scalar, vector, and tensor) which must be defined by a constitutive relation can be represented as a function of these variables. Therefore, Ψ takes the form

$$\Psi = \Psi(\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T, T, \rho), \tag{15}$$

where the assumption of a medium with purely instantaneous response has been also made [17]. Similar equations with the same arguments are valid for the stress tensor, electric conduction current vector, etc.

Substitution of Eq. (15) into Eq. (14) yields an implication of the following restrictions on the constitutive equations:

$$\frac{\partial \Psi}{\partial \mathbf{d}} = 0, \quad \frac{\partial \Psi}{\partial \nabla T} = 0, \quad s = -\frac{\partial \Psi}{\partial T}, \quad \mathbf{P} = -\rho \frac{\partial \Psi}{\partial \hat{\mathbf{E}}}, \quad \hat{\mathbf{M}} = -\rho \frac{\partial \Psi}{\partial \mathbf{B}}, \tag{16}$$

$$\rho \gamma = \boldsymbol{\tau} : \mathbf{d} + \mathbf{J}_c \cdot \hat{\mathbf{E}} + T^{-1} \mathbf{q} \cdot \nabla T \geqslant 0, \tag{17}$$

where the deviator part of stress tensor and the modified hydrostatic pressure are defined by

$$\mathbf{t} = -\varphi \mathbf{I} + \tau, \quad \varphi = \rho^2 \frac{\partial \Psi}{\partial \rho}. \tag{18}$$

The first two terms of Eq. (16) imply that Ψ is independent of $\mathbf{d}$ and ∇T. Then, the only possible dependence of the free energy upon $\hat{\mathbf{E}}$ and $\mathbf{B}$ is through its scalar invariants defined by

$$I_1 = \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}, \quad I_2 = \mathbf{B} \cdot \mathbf{B}, \quad I_3 = (\hat{\mathbf{E}} \cdot \mathbf{B})^2. \tag{19}$$

Consequently, the general form of free energy (Eq. (15)) changes into

$$\Psi = \Psi(I_1, I_2, I_3, T, \rho). \tag{20}$$

From Eqs. (16) and (20) the polarization vector and magnetomotive intensity vector can be found as

$$\mathbf{P} = -2\rho\left(\frac{\partial \Psi}{\partial I_1}\hat{\mathbf{E}} + \frac{\partial \Psi}{\partial I_3}(\hat{\mathbf{E}} \cdot \mathbf{B})\mathbf{B}\right) \equiv \varepsilon_0 \chi_c \hat{\mathbf{E}} + \lambda(\hat{\mathbf{E}} \cdot \mathbf{B})\mathbf{B}, \tag{21}$$

$$\hat{\mathbf{M}} = -2\rho\left(\frac{\partial \Psi}{\partial I_2}\mathbf{B} + \frac{\partial \Psi}{\partial I_3}(\hat{\mathbf{E}} \cdot \mathbf{B})\hat{\mathbf{E}}\right) \equiv \frac{\chi_B}{\mu_0}\mathbf{B} + \lambda(\hat{\mathbf{E}} \cdot \mathbf{B})\hat{\mathbf{E}}. \tag{22}$$

Here, notice that χ_c, χ_B, and λ are general functions of I_1–I_3 and T, ρ.

Using a similar reasoning, the symmetric deviator stress tensor for a non-linear fluid, which is defined by Eq. (18), can be expressed as (similar to that in Ref. [12], pp. 177–178)

$$\begin{aligned}
\tau = {} & \alpha_0 \mathbf{I} + \alpha_1 \mathbf{d} + \alpha_2 \mathbf{d}^2 + \alpha_3 \hat{\mathbf{E}} \otimes \hat{\mathbf{E}} + \alpha_4 \mathbf{B} \otimes \mathbf{B} \\
& + \alpha_5 \nabla T \otimes \nabla T + \alpha_6 (\hat{\mathbf{E}} \otimes \mathbf{d} \cdot \hat{\mathbf{E}})_S \\
& + \alpha_7 (\hat{\mathbf{E}} \otimes \mathbf{d}^2 \cdot \hat{\mathbf{E}})_S + \alpha_8 (\nabla T \otimes \mathbf{d} \cdot \nabla T)_S \\
& + \alpha_9 (\nabla T \otimes \mathbf{d}^2 \cdot \nabla T)_S + \alpha_{10} (\mathbf{d} \cdot \mathbf{W})_S \\
& + \alpha_{11} \mathbf{W} \cdot \mathbf{d} \cdot \mathbf{W} + \alpha_{12} (\mathbf{d}^2 \cdot \mathbf{W})_S \\
& + \alpha_{13} (\mathbf{W} \cdot \mathbf{d} \cdot \mathbf{W}^2)_S + \alpha_{14} (\hat{\mathbf{E}} \otimes \nabla T)_S \\
& + \alpha_{15} (\mathbf{W} \cdot \hat{\mathbf{E}} \otimes \mathbf{W} \cdot \hat{\mathbf{E}}) + \alpha_{16} (\hat{\mathbf{E}} \otimes \mathbf{W} \cdot \hat{\mathbf{E}})_S \\
& + \alpha_{17} (\mathbf{W} \cdot \hat{\mathbf{E}} \otimes \mathbf{W}^2 \cdot \hat{\mathbf{E}})_S \\
& + \alpha_{18} (\mathbf{W} \cdot \nabla T) \otimes (\mathbf{W} \cdot \nabla T) + \alpha_{19} \nabla T \otimes (\mathbf{W} \cdot \nabla T) \\
& + \alpha_{20} ((\mathbf{W} \cdot \nabla T) \otimes (\mathbf{W}^2 \cdot \nabla T))_S \\
& + \alpha_{21} (\mathbf{d} \cdot (\hat{\mathbf{E}} \otimes \nabla T - \nabla T \otimes \hat{\mathbf{E}}))_S \\
& + \alpha_{22} (\mathbf{W} \cdot (\hat{\mathbf{E}} \otimes \nabla T - \nabla T \otimes \hat{\mathbf{E}}))_S.
\end{aligned} \tag{23}$$

In this expression, $\mathbf{W}$ is a skew-symmetric tensor defined by, $W_{kl} = \varepsilon_{klm} B_m$, and the subscript S means the symmetric part of a dyadic. Electric conduction current and conduction heat flux in the most general case can be expressed as

$$\begin{aligned}
\mathbf{J}_c = {} & \sigma_1 \hat{\mathbf{E}} + \sigma_2 \mathbf{d} \cdot \hat{\mathbf{E}} + \sigma_3 \mathbf{d}^2 \cdot \hat{\mathbf{E}} + \sigma_4 \nabla T \\
& + \sigma_5 \mathbf{d} \cdot \nabla T + \sigma_6 \mathbf{d}^2 \cdot \nabla T + \sigma_7 \hat{\mathbf{E}} \times \mathbf{B} \\
& + \sigma_8 \nabla T \times \mathbf{B} + \sigma_9 (\mathbf{d} \cdot (\hat{\mathbf{E}} \times \mathbf{B}) - (\mathbf{d} \cdot \hat{\mathbf{E}}) \times \mathbf{B}) \\
& + \sigma_{10} (\mathbf{B} \cdot \hat{\mathbf{E}})\mathbf{B} + \sigma_{11} (\mathbf{B} \cdot \nabla T)\mathbf{B} \\
& + \sigma_{12} (\mathbf{d} \cdot (\nabla T \times \mathbf{B}) - (\mathbf{d} \cdot \nabla T)\mathbf{B}),
\end{aligned} \tag{24}$$

$$\begin{aligned}
\mathbf{q} = {} & \kappa_1 \nabla T + \kappa_2 \mathbf{d} \cdot \nabla T + \kappa_3 \mathbf{d}^2 \cdot \nabla T + \kappa_4 \hat{\mathbf{E}} \\
& + \kappa_5 \mathbf{d} \cdot \hat{\mathbf{E}} + \kappa_6 \mathbf{d}^2 \cdot \hat{\mathbf{E}} + \kappa_7 \nabla T \times \mathbf{B} \\
& + \kappa_8 \hat{\mathbf{E}} \times \mathbf{B} + \kappa_9 (\mathbf{d} \cdot (\nabla T \times \mathbf{B}) - (\mathbf{d} \cdot \nabla T) \times \mathbf{B}) \\
& + \kappa_{10} (\mathbf{B} \cdot \nabla T)\mathbf{B} + \kappa_{11} (\mathbf{B} \cdot \hat{\mathbf{E}})\mathbf{B} \\
& + \kappa_{12} (\mathbf{d} \cdot (\hat{\mathbf{E}} \times \mathbf{B}) - (\mathbf{d} \cdot \hat{\mathbf{E}}) \times \mathbf{B}),
\end{aligned} \tag{25}$$

respectively. All of the physical properties of the media in Eqs. (23)–(25) are functions of the scalar invariants and thermodynamic states. Specifically,

$$\alpha_i = \alpha_i(I_n, T, \rho), \quad \sigma_j = \sigma_j(I_n, T, \rho), \quad \kappa_j = \kappa_j(I_n, T, \rho) \quad (i = 0, 1, \ldots, 22,\ j = 1, 2, \ldots, 12,\ n = 1, 2, \ldots, 27). \tag{26}$$

The irreducible set of joint scalar invariants, which can be constructed from $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T$, including the three already defined by Eq. (19), consists of 27 invariants:

$I_4 = \mathrm{tr}(\mathbf{d}) = \nabla \cdot \mathbf{v}, \quad I_5 = \mathrm{tr}(\mathbf{d}^2), \quad I_6 = \mathrm{tr}(\mathbf{d}^3),$
$I_7 = \nabla T \cdot \nabla T, \quad I_8 = \hat{\mathbf{E}} \cdot \nabla T, \quad I_9 = \hat{\mathbf{E}} \cdot \mathbf{d} \cdot \hat{\mathbf{E}},$
$I_{10} = \hat{\mathbf{E}} \cdot \mathbf{d}^2 \cdot \hat{\mathbf{E}}, \quad I_{11} = \nabla T \cdot \mathbf{d} \cdot \nabla T,$
$I_{12} = \nabla T \cdot \mathbf{d}^2 \cdot \nabla T, \quad I_{13} = \mathbf{B} \cdot \mathbf{d} \cdot \mathbf{B},$
$I_{14} = \mathbf{B} \cdot \mathbf{d}^2 \cdot \mathbf{B}, \quad I_{15} = \mathbf{B} \cdot ((\mathbf{d} \cdot \mathbf{B}) \times (\mathbf{d}^2 \cdot \mathbf{B})),$
$I_{16} = (\mathbf{B} \cdot \nabla T)^2, \quad I_{17} = \hat{\mathbf{E}} \cdot \mathbf{d} \cdot \nabla T,$
$I_{18} = \hat{\mathbf{E}} \cdot \mathbf{d}^2 \cdot \nabla T, \quad I_{19} = \hat{\mathbf{E}} \cdot (\nabla T \times \mathbf{B}),$

$$I_{20} = (\hat{\mathbf{E}}\cdot\mathbf{B})(\nabla T\cdot\mathbf{B}),\quad I_{21} = \hat{\mathbf{E}}\cdot(\mathbf{B}\times(\mathbf{d}\cdot\hat{\mathbf{E}})),$$
$$I_{22} = \nabla T\cdot(\mathbf{B}\times(\mathbf{d}\cdot\nabla T)),\quad I_{23} = \hat{\mathbf{E}}\cdot(\mathbf{B}\times(\mathbf{d}^2\cdot\mathbf{E})),$$
$$I_{24} = \nabla T\cdot(\mathbf{B}\times(\mathbf{d}^2\cdot\nabla T)),$$
$$I_{25} = (\hat{\mathbf{E}}\cdot\mathbf{B})\hat{\mathbf{E}}\cdot(\mathbf{B}\times(\mathbf{d}\cdot\mathbf{B})),$$
$$I_{26} = (\nabla T\cdot\mathbf{B})\nabla T\cdot(\mathbf{B}\times(\mathbf{d}\cdot\mathbf{B})),$$
$$I_{27} = \hat{\mathbf{E}}\cdot(\mathbf{B}\times(\mathbf{d}\cdot\nabla T)) + \nabla T\cdot(\mathbf{B}\times(\mathbf{d}\cdot\hat{\mathbf{E}})). \tag{27}$$

3.2. Second-order theory of constitutive equations

Because so many joint invariants are involved in the constitutive equations for the stress tensor, conduction current, and heat flux vector, they cannot be used in the general form for practical applications. While the linear theory is relatively simple [6], it is inconsistent and inappropriate for cases where non-linear and/or cross effects are important. The typical examples are found in electrorheological (ER) or magnetorheological (MR) fluids. For the ER case, there is a recent continuum mechanical treatment by Rajagopal and Růžička [18], who developed a model for the ER materials within the very general framework of an electromechanical theory.

A fully consistent non-linear combined EMHD model, with a somewhat reduced complexity, may be called a second-order theory. The underlying assumption is that the electromagnetic fields, rate of strain, and temperature gradient are relatively small. More precisely, the following two assumptions will be made in the constitutive equations. First, only the terms up to second order in $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T$ will be retained. Second, terms of second order and higher in $\mathbf{d}$ will be neglected as in the case of conventional Newtonian fluids. The application of these assumptions to the general form of the previous section causes the constitutive equations for $\Psi, \mathbf{P}, \hat{\mathbf{M}}$, and φ to simplify as follows:

$$\Psi = \Psi(I_1, I_2, T, \rho) = \Psi_0 - \frac{1}{2\rho}\left(\varepsilon_0\chi_e I_1 + \frac{\chi_B}{\mu_0}I_2\right), \tag{28}$$

$$\varphi = \rho^2\frac{\partial\Psi_0}{\partial\rho} - \frac{\varepsilon_0\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{\chi_e}{\rho}\right)I_1 - \frac{\rho^2}{2\mu_0}\frac{\partial}{\partial\rho}\left(\frac{\chi_B}{\rho}\right)I_2$$
$$\equiv \varphi_0 + \varphi_e + \varphi_m, \tag{29}$$

$$\mathbf{P} = \varepsilon_0\chi_e\hat{\mathbf{E}} \equiv \varepsilon_p\hat{\mathbf{E}},\quad \hat{\mathbf{M}} = \frac{\chi_B}{\mu_0}\mathbf{B} = \frac{\mathbf{B}}{\mu_m}. \tag{30}$$

This indicates a medium with purely instantaneous response [17]. Here, Ψ_0, χ_e, χ_B and $\varphi_0, \varphi_e/I_1$, φ_m/I_2 depend on T and ρ only. By the same reasoning, the deviator part of the Cauchy stress tensor, electric conduction current, and conduction heat flux can be consistently simplified as follows:

$$\tau = (\alpha_{00} + \alpha_{01}I_1 + \alpha_{02}I_2 + \alpha_{04}I_4 + \alpha_{07}I_7 + \alpha_{08}I_8)\mathbf{I} + \alpha_1\mathbf{d} + \alpha_3\hat{\mathbf{E}}\otimes\hat{\mathbf{E}} + \alpha_4\mathbf{B}\otimes\mathbf{B} + \alpha_5\nabla T\otimes\nabla T + \alpha_{10}(\mathbf{d}\cdot\mathbf{W})_s + \alpha_{14}(\hat{\mathbf{E}}\otimes\nabla T)_s, \tag{31}$$

$$\mathbf{J}_c = (\sigma_1 + \sigma_{1b}I_4)\hat{\mathbf{E}} + \sigma_2\mathbf{d}\cdot\hat{\mathbf{E}} + (\sigma_4 + \sigma_{4b}I_4)\nabla T + \sigma_5\mathbf{d}\cdot\nabla T + \sigma_7\hat{\mathbf{E}}\times\mathbf{B} + \sigma_8\nabla T\times\mathbf{B}, \tag{32}$$

$$\mathbf{q} = (\kappa_1 + \kappa_{1b}I_4)\nabla T + \kappa_2\mathbf{d}\cdot\nabla T + (\kappa_4 + \kappa_{4b}I_4)\hat{\mathbf{E}} + \kappa_5\mathbf{d}\cdot\hat{\mathbf{E}} + \kappa_7\nabla T\times\mathbf{B} + \kappa_8\hat{\mathbf{E}}\times\mathbf{B}. \tag{33}$$

Here, all coefficients denoted with α, σ, κ are general functions of T and ρ. Note that Eq. (18) still holds with Eqs. (29) and (31).

The possible restrictions imposed on the coefficients can be sought through the C–D inequality. First, one of the immediate implication of Eq. (17) is

$$\tau = 0 \text{ when } \mathbf{d} = 0,\quad \hat{\mathbf{E}} = 0, \nabla T = 0. \tag{34}$$

From Eq. (31) it follows that this implies

$$\alpha_{00} = \alpha_{02} = 0,\quad \alpha_4 = 0. \tag{35}$$

Hence, when $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}$ and ∇T are not zero, the non-negative irreversible entropy generation function can be computed as

$$\begin{aligned}\rho\gamma = {} & \{\alpha_1\mathbf{d}:\mathbf{d} + \alpha_{04}(\mathrm{tr}(\mathbf{d}))^2\} + \{\sigma_1(\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}) \\ & + (T^{-1}\kappa_4 + \sigma_4)(\hat{\mathbf{E}}\cdot\nabla T) + T^{-1}\kappa_1(\nabla T\cdot\nabla T)\} \\ & + (\alpha_3 + \sigma_2)(\hat{\mathbf{E}}\cdot\mathbf{d}\cdot\hat{\mathbf{E}}) + (\alpha_5 + T^{-1}\kappa_2) \\ & \times(\nabla T\cdot\mathbf{d}\cdot\nabla T) + \alpha_{10}(\mathbf{d}\cdot\mathbf{W})_s:\mathbf{d} \\ & + \{(\alpha_{01} + \sigma_{1b})(\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}) + (\alpha_{08} + T^{-1}\kappa_{4b} \\ & + \sigma_{4b})(\hat{\mathbf{E}}\cdot\nabla T) + (\sigma_{07} + T^{-1}\kappa_{1b}) \\ & \times(\nabla T\cdot\nabla T)\}\mathrm{tr}(\mathbf{d}) + (\alpha_{14} + T^{-1}\kappa_5 + \sigma_5) \\ & \times(\hat{\mathbf{E}}\cdot\mathbf{d}\cdot\nabla T) + (T^{-1}\kappa_8 - \sigma_8)\nabla T\cdot(\hat{\mathbf{E}}\times\mathbf{B}).\end{aligned} \tag{36}$$

Note that the two expressions in braces of the first line are quadratic in $\mathbf{d}$ and $\hat{\mathbf{E}}, \nabla T$, while each of the remaining scalar terms are cubic in $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T$. In order that this function should be non-negative definite for any values of $\mathbf{d}, \hat{\mathbf{E}}, \mathbf{B}, \nabla T$, the following relations among the physical properties must be satisfied [12,13]:

$$\begin{aligned}
&\alpha_1 \geqslant 0, \quad 3\alpha_{04} + \alpha_1 \geqslant 0, \quad \sigma_1 \geqslant 0, \quad \kappa_1 \geqslant 0,\\
&(T^{-1}\kappa_4 + \sigma_4)^2 \leqslant 4T^{-1}\kappa_1\sigma_1,\\
&\alpha_3 = -\sigma_2, \quad \alpha_5 = -T^{-1}\kappa_2, \quad \alpha_{10} = 0,\\
&\alpha_{01} = -\sigma_{1b}, \quad \alpha_{08} = -T^{-1}\kappa_{4b} - \sigma_{4b},\\
&\alpha_{07} = -T^{-1}\kappa_{1b}, \quad \alpha_{14} = -T^{-1}\kappa_5 - \sigma_5,\\
&\sigma_8 = T^{-1}\kappa_8.
\end{aligned} \tag{37}$$

Therefore, the final expression for the deviator part of the Cauchy stress tensor simplifies to

$$\begin{aligned}
\boldsymbol{\tau} = &-(\sigma_{1b}I_1 - \alpha_{04}I_4 + T^{-1}\kappa_{1b}I_7\\
&+ (T^{-1}\kappa_{4b} + \sigma_{4b})I_{15})\mathbf{I} + \alpha_1\mathbf{d} - \sigma_2\hat{\mathbf{E}}\otimes\hat{\mathbf{E}}\\
&- T^{-1}\kappa_2\nabla T\otimes\nabla T - (T^{-1}\kappa_5 + \sigma_5)(\hat{\mathbf{E}}\otimes\nabla T)_S.
\end{aligned} \tag{38}$$

There is a slight change in the expression for the electric conduction current from Eq. (32) into

$$\begin{aligned}
\mathbf{J}_c = &(\sigma_1 + \sigma_{1b}I_4)\hat{\mathbf{E}} + \sigma_2\mathbf{d}\cdot\hat{\mathbf{E}} + (\sigma_4 + \sigma_{4b}I_4)\nabla T\\
&+ \sigma_5\mathbf{d}\cdot\nabla T + \sigma_7\hat{\mathbf{E}}\times\mathbf{B} + T^{-1}\kappa_8\nabla T\times\mathbf{B},
\end{aligned} \tag{39}$$

while the conduction heat flux, Eq. (33), remains unchanged.

3.3. Constitutive equations for incompressible flows

In the case of an incompressible fluid, all terms that contain $I_4 = (= \mathrm{tr}(\mathbf{d}) = \nabla\cdot\mathbf{v})$ reduce to zero. This means that in electromagneto hydrodynamics (EMHD) there is no dependence of physical properties upon density. The equation for the modified hydrostatic pressure φ can be written as

$$\begin{aligned}
\varphi &= \rho^2\frac{\partial\Psi_0}{\partial\rho} + \frac{\varepsilon_0\chi_e}{2}(\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}) + \frac{\chi_B}{2\mu_0}(\mathbf{B}\cdot\mathbf{B})\\
&\equiv p + p_e + p_m,
\end{aligned} \tag{40}$$

while the expressions for the polarization and magnetization (Eq. (30)) remain unchanged. Similarly, the deviator part of the stress tensor and conduction vectors reduce from Eqs. (38), (39) and (33) to

$$\begin{aligned}
\boldsymbol{\tau} = &2\mu_v\mathbf{d} - \sigma_2\hat{\mathbf{E}}\otimes\hat{\mathbf{E}} - T^{-1}\kappa_2\nabla T\otimes\nabla T\\
&- (T^{-1}\kappa_5 + \sigma_5)(\hat{\mathbf{E}}\cdot\nabla T)_S,
\end{aligned} \tag{41}$$

$$\begin{aligned}
\mathbf{J}_c = &\sigma_1\hat{\mathbf{E}} + \sigma_2\mathbf{d}\cdot\hat{\mathbf{E}} + \sigma_4\nabla T + \sigma_5\mathbf{d}\cdot\nabla T\\
&+ \sigma_7\hat{\mathbf{E}}\times\mathbf{B} + T^{-1}\kappa_8\nabla T\times\mathbf{B},
\end{aligned} \tag{42}$$

$$\begin{aligned}
\mathbf{q} = &\kappa_1\nabla T + \kappa_2\mathbf{d}\cdot\nabla T + \kappa_4\hat{\mathbf{E}} + \kappa_5\mathbf{d}\cdot\hat{\mathbf{E}}\\
&+ \kappa_7\nabla T\times\mathbf{B} + \kappa_8\hat{\mathbf{E}}\times\mathbf{B}.
\end{aligned} \tag{43}$$

Here, the coefficient of fluid viscosity is defined as $\mu_v = \alpha_1/2$. Notice that in the EMHD, the physical properties of the media, χ_e, χ_B; μ_v; $\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7$; $\kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_8$, can be either constants or functions of temperature only. The pressure, p, must be determined such that the incompressibility condition, $\nabla\cdot\mathbf{v} = 0$, should be satisfied everywhere in the flow-field. The magnetic field intensity $\mathbf{H}$ and the total magnetization vector $\mathbf{M}$ can be readily determined by substituting Eq. (30) into Eq. (6).

4. Non-linear EMHD model

4.1. Governing equations

A full system of governing equations for the incompressible flows under the combined effect of electromagnetic forces is described in this section by using the constitutive equations which have been derived through the second-order theory.

A slight modification, which is called the Boussinesq approximation, of the conservation laws is needed to be compatible with incompressible flows. In the Boussinesq approximation, the variation of density is kept only in the gravity force of the momentum equation. A linear dependence of density on temperature is assumed there. If the thermal buoyancy is the only mechanical body force acting on the fluid, the linear momentum equation can be rewritten as

$$\frac{D\mathbf{v}}{Dt} = \rho^{-1}(\nabla\cdot\mathbf{t} + \mathbf{F}^{em}) - g[1 - \alpha(T - T_0)]\mathbf{i}_3. \tag{44}$$

Here, α is the coefficient of thermal expansion of the fluid, and $\mathbf{i}_3$ is the unit vector directing vertically upward. It is a common practice to neglect the thermal buoyancy, and to use $\alpha = 0$ in forced convection studies. For incompressible flows, the energy conservation (Eq. (10)) can be rewritten in terms of temperature, T, as

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \mathbf{t}:\mathbf{d} + Q_h + \nabla\cdot\mathbf{q} + \hat{\mathbf{E}}\cdot\frac{D\mathbf{P}}{Dt} - \hat{\mathbf{M}}\cdot\frac{D\mathbf{B}}{Dt} + \mathbf{J}_c\cdot\hat{\mathbf{E}}. \tag{45}$$

By substituting the relevant constitutive equations into all balance laws, the following system of governing equations can be obtained.

Maxwell's equations (Eqs. (1)–(4)):

$$\nabla\cdot(\varepsilon_0\mathbf{E} + \varepsilon_p\hat{\mathbf{E}}) = q_e, \tag{46}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{47}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{48}$$

$$\nabla\times\left(\frac{\mathbf{B}}{\mu} + \varepsilon_p\mathbf{v}\times\hat{\mathbf{E}}\right) = \frac{\partial}{\partial t}(\varepsilon_0\mathbf{E} + \varepsilon_p\hat{\mathbf{E}}) + q_e\mathbf{v} + \sigma_1\hat{\mathbf{E}} + \sigma_2\mathbf{d}\cdot\hat{\mathbf{E}} + \sigma_4\nabla T + \sigma_5\mathbf{d}\cdot\nabla T + \sigma_7\hat{\mathbf{E}}\times\mathbf{B} + T^{-1}\kappa_8\nabla T\times\mathbf{B}. \tag{49}$$

Navier–Stokes equations:

$$\nabla\cdot\mathbf{v} = 0. \tag{50}$$

$$\begin{aligned}\rho\frac{D\mathbf{v}}{Dt} = &-\rho g[1 - \alpha(T - T_0)]\mathbf{i}_3 - \nabla(p + p_e + p_m) \\ &+ \nabla\cdot(\mu_v(\nabla\mathbf{v} + \nabla\mathbf{v}^t)) - \nabla\cdot(\sigma_2(\hat{\mathbf{E}}\otimes\hat{\mathbf{E}})) \\ &- \nabla\cdot(T^{-1}\kappa_2(\nabla T\otimes\nabla T)) - \nabla\cdot((T^{-1}\kappa_5 \\ &+ \sigma_5)(\hat{\mathbf{E}}\otimes\nabla T)_S) + q_e\hat{\mathbf{E}} + \sigma_1\hat{\mathbf{E}}\times\mathbf{B} \\ &+ \sigma_2\mathbf{d}\cdot\hat{\mathbf{E}}\times\mathbf{B} + \sigma_4\nabla T\times\mathbf{B} + \sigma_5\mathbf{d}\cdot\nabla T\times\mathbf{B} \\ &+ \sigma_7(\hat{\mathbf{E}}\times\mathbf{B})\times\mathbf{B} + T^{-1}\kappa_8(\nabla T\times\mathbf{B})\times\mathbf{B} \\ &+ \varepsilon_p(\nabla\mathbf{E})\cdot\hat{\mathbf{E}} + (\nabla\mathbf{B})\cdot\left(\frac{\mathbf{B}}{\mu_m} - \varepsilon_p\mathbf{v}\times\hat{\mathbf{E}}\right) \\ &+ \frac{D}{Dt}(\varepsilon_p(\hat{\mathbf{E}}\times\mathbf{B})).\end{aligned} \tag{51}$$

$$\begin{aligned}\rho C_p\frac{DT}{Dt} = &\, Q_h + \nabla\cdot(\kappa_1\nabla T + \kappa_2\mathbf{d}\cdot\nabla T + \kappa_4\hat{\mathbf{E}} \\ &+ \kappa_5\mathbf{d}\cdot\hat{\mathbf{E}} + \kappa_7\nabla T\times\mathbf{B} + \kappa_8\hat{\mathbf{E}}\cdot\mathbf{B}) \\ &+ \sigma_1\hat{\mathbf{E}}\cdot\hat{\mathbf{E}} + \sigma_4\hat{\mathbf{E}}\cdot\nabla T - \frac{\kappa_2}{T}\nabla T\cdot\mathbf{d}\cdot\nabla T \\ &- \frac{\kappa_5}{T}\hat{\mathbf{E}}\cdot\mathbf{d}\cdot\nabla T + \frac{\kappa_8}{T}\hat{\mathbf{E}}\cdot(\nabla T\times\mathbf{B}) \\ &+ \hat{\mathbf{E}}\cdot\frac{D(\varepsilon_p\hat{\mathbf{E}})}{Dt} - \frac{\mathbf{B}}{\mu_m}\cdot\frac{D\mathbf{B}}{Dt}.\end{aligned} \tag{52}$$

In the above system, the equation of electric charge conservation is omitted because it can be readily obtained by combining the first and fourth of the Maxwell's equations. The viscous dissipation term and the unsteady pressure term on the right side of the energy conservation (Eq. (52)) have been neglected, as is usually done in incompressible viscous flow modeling [19].

4.2. Non-dimensionalization and dimensionless parameters

Because of the extreme complexity of the complete, non-linear, fully coupled EMHD model, it is practically impossible at the present time to contemplate development of a numerical simulation package for its integration. Consequently, the complete general EMHD model should be simplified for particular circumstances. This simplification must be performed in a consistent manner starting from the general EMHD model. Elimination of certain terms could be justified by an order of magnitude analysis that is most efficiently accomplished by performing a complete non-dimensionalization of the model.

Thus all flow-field, electric field, and magnetic field parameters and all physical properties will be non-dimensionalized. The reference length and velocity are denoted by L and V, respectively. The reference time will be defined by the ratio, L/V. The reference values of the pressure, electric field, and magnetic flux are taken to be $\rho V^2, E_0$, and B_0, respectively. Temperature, however, is non-dimensionalized by the equation

$$\theta = \frac{T - T_0}{\Delta T}. \tag{53}$$

Additional reference values for each parameter will be designated with a subscript R. The field variables and material properties with a prime ′ denote the corresponding non-dimensionalized values. For example, the magnetic flux **B** and the electric permittivity ε can be represented as $\mathbf{B} = B_0 \mathbf{B}'$ and $\varepsilon = \varepsilon_R \varepsilon'$.

Non-dimensional forms of the governing equations (Eqs. (46)–(52)) for the complete EMHD model are presented next, followed by a set of dimensionless parameters. There is no restriction to the temporal and spatial variation of the material properties, such as conduction coefficients, electric permittivity, and coefficient of viscosity.

A very important parameter that will be helpful in the elimination of a number of terms in the EMHD model is the squared ratio of the reference fluid speed and the speed of light, that is, square of the electromagnetic Mach number,

$$\beta = \varepsilon_0 \mu_0 V^2 = \frac{V^2}{c^2} \ll 1. \tag{54}$$

This parameter assumes an extremely small value for incompressible flows. For example, $\beta \approx 10^{-14}$ when $V = 20$ m/s. This is the typical upper bound for the speed of an aqueous solution at room temperature and atmospheric pressure, in order to avoid cavitation.

Also, two additional dimensionless quantities will be introduced for compactness,

$$\mathbf{e} = \mathbf{E}' + N_B \mathbf{v}' \times \mathbf{B}', \quad \Theta = 1 + N_T \theta. \tag{55}$$

Then, the non-dimensionalized Maxwell's equations for EMHD model are

$$\nabla' \cdot (\varepsilon' \mathbf{E}') + N_B \bar{\varepsilon}_p \nabla' \cdot (\varepsilon'_p (\mathbf{v}' \times \mathbf{B}')) = N_q q'_e, \tag{56}$$

$$\nabla' \cdot \mathbf{B}' = 0, \tag{57}$$

$$\nabla' \times \mathbf{E}' = - N_B \frac{\partial \mathbf{B}'}{\partial t'}, \tag{58}$$

$$\begin{aligned} & N_B \nabla' \times \frac{\mathbf{B}'}{\mu'} + \beta \mu_{r_R} \chi_{e_R} \nabla' \times (\varepsilon'_p (\mathbf{v}' \times \mathbf{e})) \\ & = \beta \mu_{r_R} \varepsilon_{r_R} \left[\frac{\partial}{\partial t'} (\varepsilon' \mathbf{E}') + N_B \bar{\varepsilon}_p \frac{\partial}{\partial t'} (\varepsilon'_p (\mathbf{v}' \times \mathbf{B}')) \right. \\ & \quad \left. + N_q (q'_e \mathbf{v}') \right] + N_{\sigma 1} (\sigma'_1 \mathbf{e}) + N_{\sigma 2} (\sigma'_2 \mathbf{d}' \cdot \mathbf{e}) \\ & \quad + N_{\sigma 4} (\sigma'_4 \nabla' \theta) + N_{\sigma 5} (\sigma'_5 \mathbf{d}' \cdot \nabla' \theta) \\ & \quad + N_{\sigma 7} (\sigma'_7 \mathbf{e} \times \mathbf{B}') + N_T \tilde{N}_{\kappa 8} \kappa'_8 \Theta^{-1} (\nabla' \theta \times \mathbf{B}'). \end{aligned} \tag{59}$$

Non-dimensionalized Navier–Stokes equations for the EMHD model are

$$\nabla' \cdot \mathbf{v}' = 0. \tag{60}$$

$$\begin{aligned} \frac{\mathrm{D}\mathbf{v}'}{\mathrm{D}t'} & = - \nabla' p' - N_{es} \nabla' \left(\frac{\varepsilon'_p (\mathbf{e} \cdot \mathbf{e})}{2} \right) - N_{ms} \nabla' \left(\frac{\mathbf{B}' \cdot \mathbf{B}'}{2 \mu'_m} \right) \\ & \quad - \mathrm{Fr}^{-2} (1 - \alpha \Delta T \theta) \mathbf{i}_3 + \frac{1}{\mathrm{Re}} \nabla' \cdot [\mu'_v (\nabla' \mathbf{v} \\ & \quad + (\nabla' \mathbf{v}')^t)] - \mathrm{K}_{\sigma 2} \nabla' \cdot (\sigma'_2 \mathbf{e} \otimes \mathbf{e}) \\ & \quad - K_{\kappa 2} N_T \nabla' \cdot (\Theta^{-1} \kappa'_2 \nabla' \theta \otimes \nabla' \theta) \\ & \quad - K_{\kappa 5} N_T \nabla' \cdot (\Theta^{-1} \kappa'_5 (\mathbf{e} \otimes \nabla' \theta)_S) \\ & \quad - K_{\sigma 5} \nabla' \cdot (\sigma'_5 (\mathbf{e} \otimes \nabla' \theta)_S) + N_{Co} (q'_e \mathbf{e}) \\ & \quad + K_{\sigma 1} (\sigma'_1 \mathbf{e} \times \mathbf{B}') + N_B K_{\sigma 2} [\sigma'_2 (\mathbf{d}' \cdot \mathbf{e}) \times \mathbf{B}'] \\ & \quad + K_{\sigma 4} (\sigma'_4 \nabla' \theta \times \mathbf{B}') + N_B K_{\sigma 5} \\ & \quad \times (\sigma'_5 (\mathbf{d}' \cdot \nabla' \theta) \times \mathbf{B}') + K_{\sigma 7} (\sigma'_7 (\mathbf{e} \times \mathbf{B}') \times \mathbf{B}') \\ & \quad + N_{\kappa 8} N_T \Theta^{-1} (\kappa'_8 (\nabla' \theta \times \mathbf{B}') \times \mathbf{B}') \\ & \quad + N_{es} \varepsilon'_p (\nabla' \mathbf{E}') \cdot \mathbf{e} + (\nabla' \mathbf{B}') \cdot \left(N_{ms} \frac{\mathbf{B}'}{\mu'_m} \right. \\ & \quad \left. - N_B N_{es} \varepsilon'_p (\mathbf{v}' \times \mathbf{e}) \right) + N_B N_{es} \frac{\mathrm{D}}{\mathrm{D}t'} (\varepsilon'_p (\mathbf{e} \times \mathbf{B}')). \end{aligned} \tag{61}$$

$$\begin{aligned} C'_p \frac{\mathrm{D}\theta}{\mathrm{D}t'} & = N_{Q_h} Q'_h + \mathrm{Pe}^{-1} \nabla' \cdot (\kappa'_1 \nabla' \theta) \\ & \quad + N_{\kappa 2} \nabla' \cdot (\kappa'_2 \mathbf{d}' \cdot \nabla' \theta) + N_{\kappa 4} \nabla' \cdot (\kappa'_4 \mathbf{e}) \\ & \quad + N_{\kappa 5} \nabla' \cdot (\kappa'_5 \mathbf{d}' \cdot \mathbf{e}) + N_{\kappa 7} \nabla' \cdot (\kappa'_7 \nabla' \theta \times \mathbf{B}') \\ & \quad + N_{\kappa 8} \nabla' \cdot (\kappa'_8 \mathbf{e} \times \mathbf{B}') + \Lambda_{\sigma 1} \sigma'_1 (\mathbf{e} \cdot \mathbf{e}) \end{aligned}$$

$$+\Lambda_{\sigma 4}\sigma_4'(\mathbf{e}\cdot\nabla'\theta) - N_{\kappa 2}N_{\mathrm{T}}\Theta^{-1}\kappa_2'$$
$$\times(\nabla'\theta\cdot\mathbf{d}'\cdot\nabla'\theta) - N_{\kappa 5}N_{\mathrm{T}}\Theta^{-1}\kappa_5'(\mathbf{e}\cdot\mathbf{d}'\cdot\nabla'\theta)$$
$$+ N_{\kappa 8}N_{\mathrm{T}}\Theta^{-1}\kappa_8'(\mathbf{e}\cdot(\nabla'\theta\times\mathbf{B}'))$$
$$+ \mathrm{Ec}N_{\mathrm{es}}\mathbf{e}\cdot\frac{\mathrm{D}(\varepsilon_{\mathrm{p}}'\mathbf{e})}{\mathrm{D}t'} - \mathrm{Ec}N_{\mathrm{ms}}\frac{\mathbf{B}'}{\mu_{\mathrm{m}}'}\cdot\frac{\mathrm{D}\mathbf{B}'}{\mathrm{D}t'}. \tag{62}$$

Note here that the electromagnetic field contains only three unknowns: $\mathbf{E}'$, $\mathbf{B}'$, and q_{e}'.

The non-dimensional parameters appearing in the system of equations (Eqs. (56)–(62)) are

$$N_{\mathrm{B}} = \frac{VB_0}{E_0},\quad N_{\mathrm{T}} = \frac{\Delta T}{T_0},\quad \bar{\varepsilon}_{\mathrm{p}} = \frac{\varepsilon_{\mathrm{p_R}}}{\varepsilon_{\mathrm{R}}},\quad N_{\mathrm{q}} = \frac{q_{\mathrm{e_R}}L}{\varepsilon_{\mathrm{R}}E_0},$$

$$N_{\sigma 1} = \sigma_{1_{\mathrm{R}}}\mu_{\mathrm{R}}VL = \text{magnetic Reynolds number},$$

$$N_{\sigma 2} = \sigma_{2_{\mathrm{R}}}\mu_{\mathrm{R}}V^2,\quad N_{\sigma 4} = \frac{\sigma_{4_{\mathrm{R}}}\mu_{\mathrm{R}}\Delta TV}{E_0},$$

$$N_{\sigma 5} = \frac{\sigma_{5_{\mathrm{R}}}\mu_{\mathrm{R}}\Delta TV^2}{E_0L},\quad N_{\sigma 7} = \sigma_{7_{\mathrm{R}}}\mu_{\mathrm{R}}B_0VL,$$

$$\tilde{N}_{\kappa 8} = \frac{\kappa_{8_{\mathrm{R}}}\mu_{\mathrm{R}}VB_0}{E_0},\quad N_{\mathrm{es}} = \frac{\varepsilon_{\mathrm{p_R}}E_0^2}{\rho V^2},$$

$$N_{\mathrm{ms}} = \frac{B_0^2}{\rho V^2\mu_{\mathrm{m_R}}},\quad \mathrm{Fr} = \frac{V}{\sqrt{gL}} = \text{Froude number},$$

$$\mathrm{Re} = \frac{\rho VL}{\mu_{\mathrm{v_R}}} = \text{Reynolds number},\quad K_{\sigma 2} = \frac{\sigma_{2_{\mathrm{R}}}E_0^2}{\rho V^2},$$

$$K_{\kappa 2} = \frac{\kappa_{2_{\mathrm{R}}}\Delta T}{\rho V^2L^2},\quad K_{\kappa 5} = \frac{\kappa_{5_{\mathrm{R}}}E_0}{\rho V^2L},\quad K_{\sigma 5} = \frac{\sigma_{5_{\mathrm{R}}}E_0\Delta T}{\rho V^2L},$$

$$N_{\mathrm{Co}} = \frac{q_{\mathrm{e_R}}E_0L}{\rho V^2} = \text{Coulomb number},$$

$$K_{\sigma 1} = \frac{\sigma_{1_{\mathrm{R}}}E_0B_0L}{\rho V^2},\quad K_{\sigma 4} = \frac{\sigma_{4_{\mathrm{R}}}B_0\Delta T}{\rho V^2},$$

$$K_{\sigma 7} = \frac{\sigma_{7_{\mathrm{R}}}E_0B_0^2}{\rho V^2},\quad N_{\kappa 8} = \frac{\kappa_{8_{\mathrm{R}}}B_0^2}{\rho V^2},$$

$$N_{\mathrm{Q_h}} = \frac{Q_{\mathrm{h_R}}L}{\rho C_{\mathrm{p_R}}\Delta TV},$$

$$\mathrm{Pe} = \frac{\rho C_{\mathrm{p_R}}VL}{\kappa_{1_{\mathrm{R}}}} = \text{Peclet number},$$

$$N_{\kappa 2} = \frac{\kappa_{2_{\mathrm{R}}}}{\rho C_{\mathrm{p_R}}L^2},\quad N_{\kappa 4} = \frac{\kappa_{4_{\mathrm{R}}}E_0}{\rho C_{\mathrm{p_R}}\Delta TV},$$

$$N_{\kappa 5} = \frac{\kappa_{5_{\mathrm{R}}}E_0}{\rho C_{\mathrm{p_R}}\Delta TL},\quad N_{\kappa 7} = \frac{\kappa_{7_{\mathrm{R}}}B_0}{\rho C_{\mathrm{p_R}}VL},$$

$$N_{\kappa 8} = \frac{\kappa_{8_{\mathrm{R}}}E_0B_0}{\rho C_{\mathrm{p_R}}\Delta TV},\quad \Lambda_{\sigma 1} = \frac{\sigma_{1_{\mathrm{R}}}E_0^2L}{\rho C_{\mathrm{p_R}}\Delta TV},$$

$$\Lambda_{\sigma} = \frac{\sigma_{4_{\mathrm{R}}}E_0}{\rho C_{\mathrm{p_R}}V},\quad \mathrm{Ec} = \frac{V^2}{C_{\mathrm{p_R}}\Delta T} = \text{Eckert number}. \tag{63}$$

5. Conclusions

Starting from very general fundamental principles and material constitutive relations, a complete analytical model (EMGD) was outlined for combined influence of unsteady electric and magnetic fields in a moving fluid that is polarizable and magnetizable. A simpler model of the EMGD was then derived by neglecting the terms which are higher than the second order in the original complete EMGD model. Finally, a complete analytical model for the incompressible flow-field under the combined influence of unsteady electric and magnetic fields (EMHD) was derived in a consistent manner. The EMHD model allows for non-linear, cross effects of electromagnetic field and clearly specifies which physical properties need to be known for the successful modeling of such flows. The EMHD model was also represented in its non-dimensional fully consistent form.

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