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Simultaneous Determination of Temperatures, Heat Fluxes, Deformations, and Tractions on Inaccessible Boundaries

A finite element method formulation for the detection of unknown steady boundary conditions in heat conduction and linear elasticity and combined thermoelasticity continuum problems is presented. The present finite element method formulation is capable of determining displacements, surface stresses, temperatures, and heat fluxes on the boundaries where such quantities are unknown or inaccessible, provided such quantities are sufficiently overspecified on other boundaries. Details of the discretization, linear system solution techniques, and sample results for two-dimensional problems are presented.

Introduction

It is often difficult and even impossible to place temperature probes, heat flux probes, or strain gauges on certain parts of a surface of a solid body. This can be either due to its small size or geometric inaccessibility or because of the hostile environment on that surface. With an appropriate inverse method these unknown boundary values can be deduced from additional information that should be made available at a finite number of points within the body or on some other surfaces of the solid body. In the case of steady thermoelasticity, the objective of an inverse boundary condition determination problem is to deduce displacements, tractions, temperatures, and heat fluxes on any surfaces or surface elements where such information is unknown. This represents a multidisciplinary (combined heat conduction and linear elasticity) inverse problem. A separate problem of inverse determination of unknown boundary conditions in steady heat conduction has been solved by a variety of methods (Larsen, 1985; Martin and Dulikravich, 1996a; Hensel and Hills, 1989). Similarly, a separate inverse boundary condition determination problem in linear elastostatics has been solved by different methods (Maniatty and Zabaras, 1994; Martin et al., 1995).

Our objective is to develop and demonstrate a novel approach for the simultaneous determination of both thermal and elasticity conditions on parts of a solid body surface. It should be pointed out that the iterative method for the solution of inverse problems to be discussed in this paper is entirely different from the noniterative approach based on boundary element method that has been used separately in linear heat conduction (Martin and Dulikravich, 1996a) and linear elasticity (Martin et al., 1995). Moreover, the current method has nothing in common with a more familiar inverse shape design problem (Kassab et al., 1994; Martin and Dulikravich, 1996b).

For inverse problems, the unknown boundary conditions on parts of the boundary can be determined by overspecifying the boundary conditions (enforcing both Dirichlet and Neumann-type boundary conditions) on at least some of the remaining portions of the boundary, and providing either Dirichlet or Neumann type boundary conditions on the rest of the boundary. It is possible, after a series of algebraic manipulations, to transform the original system of equations into a system which enforces the overspecified boundary conditions and includes the unknown boundary conditions as a part of the unknown solution vector. This formulation is an adaptation of a method by Martin and Dulikravich (1996a) for the inverse detection of boundary conditions in steady heat conduction and by Martin et al. (1995) for finding unknown boundary tractions and deformations in elastostatics using the boundary element method.

The main novelty of the current method is that it is capable of treating heat conduction and linear elasticity simultaneously (Dennis and Dulikravich, 1998a, b). Specifically, it represents an extension of the conceptual work presented by the authors (Dennis and Dulikravich, 1998a) by adding several regularization formulations. It also represents a more complete version of the work presented recently (Dennis and Dulikravich, 1998b).

Analytical and Numerical Formulation

When analyzing steady-state elasticity problems, either displacement vectors or surface traction vectors are specified everywhere on the boundary of the object. This way, one of these quantities is known, while the other is unknown at every point on the boundary.

When performing an inverse evaluation of the steady-state elasticity problem, both displacement vectors and surface traction vectors must be specified on a part of the body surface, while both are unknown on another part of the surface. Elsewhere on the body surface, either displacement vectors or surface traction vectors should be provided. The surface section where both displacement vectors and surface traction vectors are specified simultaneously is called the overspecified boundary.

Finite Element Method Formulation for Thermoelasticity. The Navier equations for linear static deformations u, v, w in three-dimensional Cartesian x, y, z coordinates are

$$(\lambda + G)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + G\nabla^2 u + X = 0 \quad (1)$$

$$(\lambda + G)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z}\right) + G\nabla^2 v + Y = 0 \quad (2)$$

$$(\lambda + G)\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2}\right) + G\nabla^2 w + Z = 0 \quad (3)$$

where

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$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}.$$

Here X, Y, Z are body forces per unit volume due to stresses from thermal expansion.

$$X = -(3\lambda + 2G)\frac{\partial \alpha \Delta\Theta}{\partial x} \tag{4}$$

$$Y = -(3\lambda + 2G)\frac{\partial \alpha \Delta\Theta}{\partial y} \tag{5}$$

$$Z = -(3\lambda + 2G)\frac{\partial \alpha \Delta\Theta}{\partial z} \tag{6}$$

This system of differential Eqs. (1)–(3) can be written in the following matrix form:

$$[L]^T([C][L]\{\delta\} - [C]\{\epsilon_0\}) - \{f\} = 0 \tag{7}$$

where the differential operator matrix, $[L]$, is defined as

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \tag{8}$$

and the elastic modulus matrix, $[C]$, is defined as

$$[C] = \frac{\lambda}{\nu}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \tag{9}$$

Casting the system of Eq. (7) in integral form using the weighted residual method yields

$$\int_\Omega [V][L]^T([C][L]\{\delta\} - [C]\{\epsilon_0\})d\Omega - \int_\Omega [V]\{f\}d\Omega = 0 \tag{10}$$

where the matrix, $[V]$, is the weight matrix which is a collection of test functions and Ω is the domain where the equations are to be solved.

$$[V] = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{bmatrix} \tag{11}$$

Equation (10) should now be integrated by parts to get the weak form of (7)

$$\int_\Omega ([L][V]^T)^T[C][L]\{\delta\}d\Omega - \int_\Omega ([L][V]^T)^T[C]\{\epsilon_0\}d\Omega - \int_\Omega [V]\{f\}d\Omega - \int_\Gamma [V]\{T\}d\Gamma = 0 \tag{12}$$

where $\{T\}$ is the vector of surface tractions on surface Γ.

$$\{T\} = [n][C][L]\{\delta\} \tag{13}$$

The matrix $[n]$ contains the Cartesian components of the unit vector normal to the surface Γ.

The displacement field in the x, y, and z-directions can now be represented with approximation functions

$$u(x, y, z) \approx \sum_{i=1}^{n} N_i(x, y, z)u_i \tag{14}$$

$$v(x, y, z) \approx \sum_{i=1}^{n} N_i(x, y, z)v_i \tag{15}$$

$$w(x, y, z) \approx \sum_{i=1}^{n} N_i(x, y, z)w_i. \tag{16}$$

Equations (14)–(16) can be rewritten in matrix form

$$\begin{Bmatrix} u(x, y, z) \\ v(u, x, y) \\ w(x, y, z) \end{Bmatrix} \approx [N]\{\delta\} \tag{17}$$

where $[N]$ is the interpolation matrix which contains the trial functions for each equation in the system. Also note that with

Nomenclature

$[C]$ = elastic modulus matrix
$[D]$ = damping matrix
E = elastic modulus of elasticity
$\{f\}$ = force vector
G = shear modulus
k = Fourier coefficient of heat conduction
$[K]$ = stiffness matrix
$[N]$ = interpolation matrix
$[n]$ = unit vector matrix
Q = heat flux
R = uniform random number between 0 and 1
S = heat source
u, v, w = deformations in the x, y, z-directions
X, Y, Z = body force components in x, y, z directions
x, y, z = Cartesian body axes
α = coefficient of thermal expansion
$\{\delta\}$ = displacement vector
$\{\epsilon_0\}$ = initial strain vector
ϵ = strain
Γ = boundary surface
λ = Lame's constant
Λ = damping parameter
ν = Poisson's ratio
$\{\sigma\}$ = stress component vector
$\bar{\sigma}$ = standard deviation
Θ = temperature
$\Delta\Theta$ = difference between local and reference temperature

Galerkin's method the weight matrix and the interpolation matrix are equal, $[N] = [V]^T$. If the matrix $[B]$ is defined as

$$[B] = [L][N] \tag{18}$$

then the substitution of the approximation functions (17) into the weak statement (12) creates the weak integral form for a finite element expressed as

$$\int_\Omega [B]^T[C][B]\{\delta\}d\Omega - \int_\Omega [B]^T[C]\{\epsilon_0\}d\Omega - \int_\Omega [N]^T\{f\}d\Omega - \int_{\Gamma_1} [N]^T\{T\}d\Gamma = 0. \tag{19}$$

This can also be written in the matrix form as

$$[K]\{\delta\} = \{f\}. \tag{20}$$

For thermal stresses, the initial elemental strain vector, ϵ_0, becomes

$$\{\epsilon_0\} = \lfloor \alpha\Delta\Theta \quad \alpha\Delta\Theta \quad \alpha\Delta\Theta \quad 0 \quad 0 \quad 0 \rfloor^T. \tag{21}$$

The local stiffness matrix, $[K]$, and the force per unit volume vector, $\{f\}$, are determined for each element in the domain and then assembled into the global system

$$[K]\{\delta\} = \{f\}. \tag{22}$$

After applying boundary conditions, the global displacements are found by solving this system of linear algebraic equations. The stresses, $\{\sigma\}$, can then be found in terms of the displacements, $\{\delta\}$, as

$$\{\sigma\} = [C][L]\{\delta\} - [C]\{\epsilon_0\}. \tag{23}$$

Finite Element Method Formulation for Thermal Problem. The temperature distribution throughout the domain can be found by solving Poisson's equation for steady linear heat conduction with a distributed steady heat source function, S, and thermal conductivity coefficient, k.

$$-k\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2}\right) = S \tag{24}$$

Applying the method of weighted residuals to (24) with a weight function, ϕ, over an element results in

$$\int_\Omega \left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2} - \frac{S}{k}\right)\phi d\Omega = 0. \tag{25}$$

Integrating this by parts once (25) creates the weak statement for an element

$$-\int_\Omega k\left(\frac{\partial\phi}{\partial x}\frac{\partial\Theta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\Theta}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\Theta}{\partial z}\right)d\Omega = \int_\Omega N_i S d\Omega - \int_\Gamma N_i(Q\cdot\hat{n})d\Omega. \tag{26}$$

Variation of the temperature across an element can be expressed by

$$\Theta(x, y, z) \approx \sum_{i=1}^{m} N_i(x, y, z)\Theta_i. \tag{27}$$

Using Galerkin's method, the weight function ϕ and the interpolation function for Θ are chosen to be the same. By defining the matrix $[E]$ as

$$[E] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_m}{\partial z} \end{bmatrix}, \tag{28}$$

the weak statement (26) for a single element can be written in the matrix form as

$$[K_c]^e\{\Theta\}^e = \{Q\}^e \tag{29}$$

where

$$[K_c]^e = \int_\Omega k[E]^T[E]d\Omega^e \tag{30}$$

$$\{Q\}^e = -\int_\Omega S\{N\}d\Omega^e + \int_\Gamma Q\{N\}d\Gamma^e. \tag{31}$$

The local stiffness matrix, $[K_c]^e$, and heat flux vector, $\{Q\}^e$, are determined for each element in the domain and then assembled into the global system

$$[K_c]\{\Theta\} = \{Q\}. \tag{32}$$

Direct and Inverse Formulations

The above equations for linear elastostatics and steady heat conduction were discretized separately by using a Galerkin's finite element method. This results in two linear systems of algebraic equations,

$$[K]\{\delta\} = \{f\}, \quad [K_c]\{\Theta\} = \{Q\}. \tag{33}$$

These systems are large, sparse, symmetric, and positive definite. Once these global systems have been formed, the boundary conditions can be applied. For a well-posed (analysis or direct) problem, the boundary conditions must be known on all boundaries of the domain. For heat conduction, either the temperature, Θ, or the heat flux, Q, must be specified at each point of the boundary. For elasticity, the displacement vector components, u_s, v_s, w_s, or the surface traction vector components, T_{x_s}, T_{y_s}, T_{z_s}, must be specified on the entire boundary.

Consider the linear system (29) for steady heat conduction on a quadrilateral finite element with boundary conditions given at points 1 and 4.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \tag{34}$$

As an example of an inverse problem, one could specify both the temperature and the heat flux at point 1, flux only at points 2 and 3, and assume the boundary conditions at point 4 as being unknown. The original system of Eq. (34) can be modified by grouping all available boundary conditions in a vector on the right-hand side

$$\begin{bmatrix} K_{12} & K_{13} & K_{14} & 0 \\ K_{22} & K_{23} & K_{24} & 0 \\ K_{32} & K_{33} & K_{34} & 0 \\ K_{42} & K_{43} & K_{44} & -1 \end{bmatrix} \begin{Bmatrix} \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 - \Theta_1 K_{11} \\ Q_2 - \Theta_1 K_{21} \\ Q_3 - \Theta_1 K_{31} \\ 0 - \Theta_1 K_{41} \end{Bmatrix}. \tag{35}$$

The same procedure can be applied to the system matrices for both steady heat conduction and elasticity in two or three dimensions. The resulting systems of equations will remain sparse, but will become unsymmetric and possibly rectangular depending on the ratio of the number of known to unknown boundary conditions. The next section will discuss techniques for solving such systems.

Regularization

Three regularization methods were applied separately to the solution of the systems of equations in attempts to increase the method's tolerance for measurement errors in the overspecified boundary conditions. Here, we consider the regularization of the inverse heat conduction problem.

The general form of a regularized system is given as (Neumaier, 1998)

$$\begin{bmatrix} K_c \\ \Lambda D \end{bmatrix} \{\Theta\} = \begin{Bmatrix} Q \\ 0 \end{Bmatrix} . \quad (36)$$

The traditional Tikhonov regularization (Tikhonov and Arsenin, 1977) is obtained when the damping matrix, $[D]$, is set equal to the identity matrix. Solving (36) in a least-squares sense minimizes the following error function:

$$\text{error}(\Theta) = \|[K_c]\{\Theta\} - \{Q\}\|_2^2 + \|\Lambda[D]\{\Theta\}\|_2^2. \quad (37)$$

This is the minimization of the residual plus a penalty term. The form of the damping matrix determines what penalty is used and the damping parameter, Λ, weights the penalty for each equation. These weights should be determined according to the error associated with the respective equation.

Method 1. This method of regularization uses a constant damping parameter Λ over the entire domain and the identity matrix as the damping matrix. This method can be considered the traditional Tikhonov method. The penalty term being minimized in this case is the square of the L_2 norm of the solution vector $\{x\}$. Minimizing this norm will tend to drive the components of $\{x\}$ to uniform values thus producing a smoothing effect. However, minimizing this penalty term will ultimately drive each component to zero, completely destroying the real solution. Thus, great care must be exercised in choosing the damping parameter Λ so that a good balance of smoothness and accuracy is achieved.

Method 2. This method of regularization uses a constant damping parameter Λ only for equations corresponding to the unknown boundary values. For all other equations $\Lambda = 0$ and $[D] = [I]$ since the largest errors occur at the boundaries where the temperatures and fluxes are unknown.

Method 3. This method uses Laplacian smoothing of the temperatures only on the boundaries where the boundary conditions are unknown. A penalty term could be constructed such that curvature of the solution on the boundary where conditions are unknown is minimized along with the residual.

$$\|\nabla^2 \Theta_{ub}\|_2^2 \rightarrow \min \quad (38)$$

Equation (38) can be discretized using the method of weighted residuals to determine the damping matrix, $[D]$.

$$\|[D]\Theta_{ub}\|_2^2 = \int_\Gamma (\nabla^2 \Theta_{ub})^2 d\Gamma = \|[K_c]\Theta_{ub}\|_2^2 \quad (39)$$

In two-dimensional planar problems, $[K_c]$ and $[D]$ can be thought of as an assembly of the linear or quadratic rod elements that discretize the boundary of the object where the boundary conditions are unknown. The main advantage of this method is its ability to smooth the solution vector without necessarily driving the components to zero and away from the true solution.

Solution of the Linear System

In general, the resulting finite element method systems for the inverse thermoelastic problems are sparse, unsymmetric, and often rectangular. These properties make the process of finding a solution to the system very challenging. Three approaches will be discussed here.

The first is to normalize the equations by multiplying both sides by the matrix transpose and solve the resulting square system with common sparse solvers.

$$[K]^T[K]\{\delta\} = [K]^T\{f\} \quad (40)$$

This approach has been found to be effective for certain inverse problems (Boschi and Fischer, 1996). The resulting normalized system is less sparse than the original system, but it is square, symmetric, and positive definite with application of regulariza-

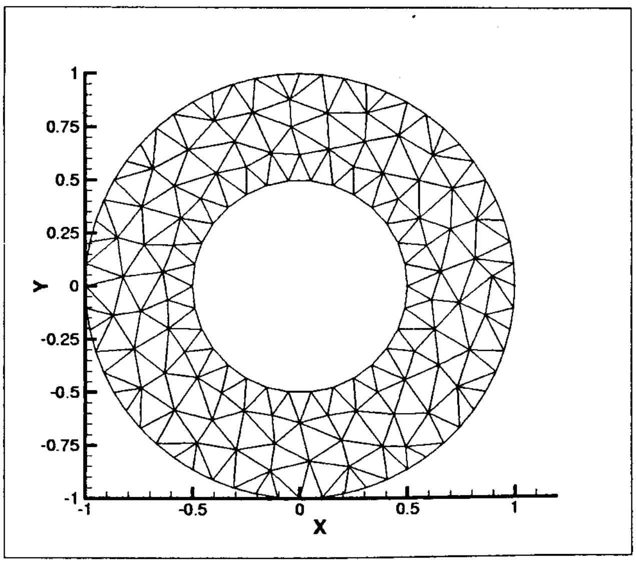

Fig. 1 Triangular mesh for an annular disk test case geometry

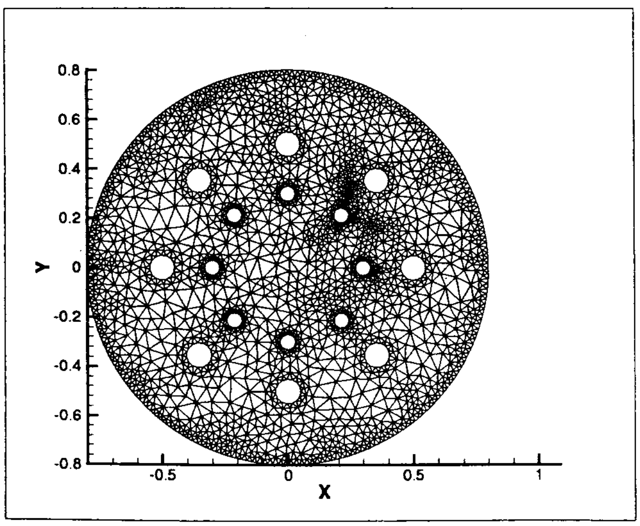

Fig. 2 Triangular mesh for a multiply connected domain test case geometry

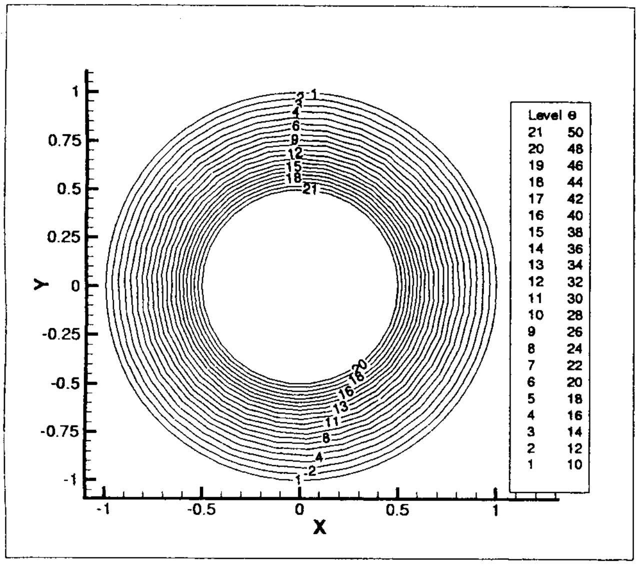

Fig. 3 Computed isotherms with inner and outer boundary temperatures specified

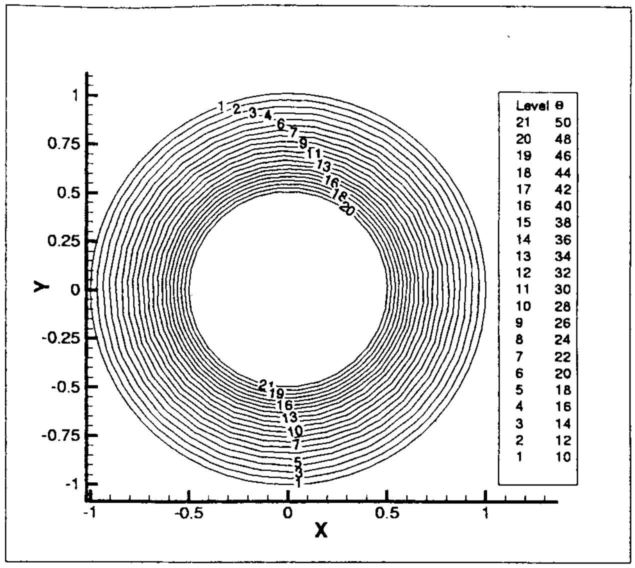

Fig. 4 Computed isotherms with outer boundary temperatures and fluxes specified. Nothing was specified on the inner boundary.

tion. The normalized system is solved with a direct method (Cholesky or LU factorization) or with an iterative method (preconditioned Krylov subspace). There are several disadvantages to this approach, among them being the computational expense of computing $[K]^T[K]$, the large in-core memory requirements, and the roundoff error incurred during the $[K]^T[K]$ multiplication.

A second approach is to use iterative methods suitable for unsymmetrical and least-squares problems. One such method is the least squares QR method, which is an extension of the well-known conjugate gradient method (Paige and Saunders, 1982). The least-squares QR method and other similar methods, such as the conjugate gradient for least squares, solve the normalized system, but without explicit computation of $[K]^T[K]$. However, convergence rates of these methods depend strongly on the condition number of the normalized system which is roughly the condition number of $[K]$ squared. Convergence can be slow when solving the systems resulting from the inverse finite element discretization since they are ill-conditioned.

A third approach is to use a noniterative method for unsymmetrical and least-squares problems such as QR factorization (Golub and Van Loan, 1996) or singular value decomposition (Golub and Van Loan, 1996). However, sparse implementations of QR or singular value decomposition solvers are needed to reduce the in-core memory requirements for the inverse finite element problems.

Numerical Results

The accuracy and efficiency of the finite element inverse formulation was tested on several simple two-dimensional problems with known analytic solutions. The method was implemented in an object-oriented finite element code written in $C++$. Elements used in the calculations were triangles with linear and quadratic interpolation functions. The triangular meshes were generated by an automatic Delauney triangulation technique (Shewchuk, 1996).

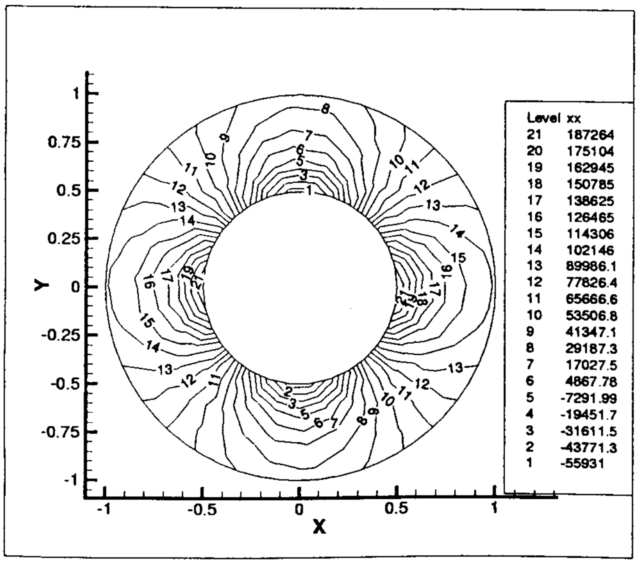

Fig. 5 Computed normal stress distribution with inner and outer boundary tractions specified

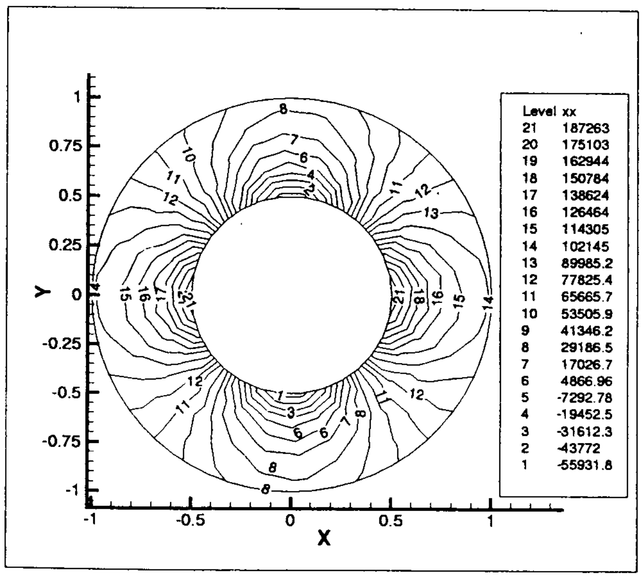

Fig. 6 Computed normal stress distribution with outer boundary tractions and displacements specified. Nothing was specified on the inner boundary.

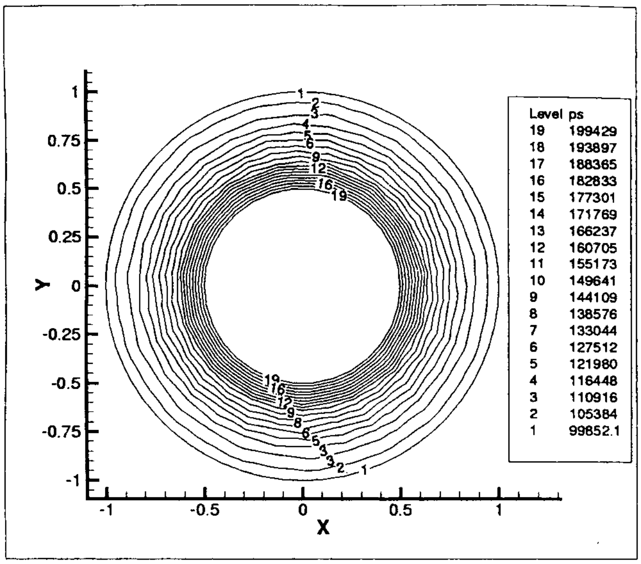

Fig. 7 Computed principal stress distribution with inner and outer boundary tractions specified

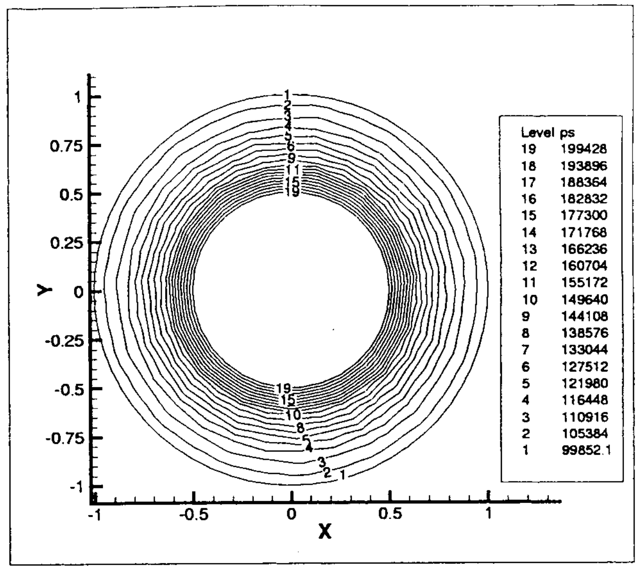

Fig. 8 Computed principal stress distribution with outer boundary tractions and displacements specified. Nothing was specified on the inner boundary.

Three different solution techniques were tested: a sparse QR factorization (Matstoms, 1991), a conjugate gradient for least squares method and least-squares QR code, and a CG solver applied to solving the normalized equations. The two basic test geometries included a rectangular plate and an annular disk (Fig. 1).

For heat conduction, one analytical test problem consisted of a rectangular homogeneous plate with uniform temperatures specified at the opposite boundaries and adiabatic conditions specified at the remaining two opposite boundaries. The finite element method solution of this direct problem was less than one percent in error compared to the analytical solution. Another simple test case was steady heat conduction in an annular homogeneous disk. In a direct (well-posed) problem a uniform temperature of 50.0 K was enforced on the inner circular boundary while a temperature of 10.0 K was enforced on the outer circular boundary. The temperature field computed with the finite element method had a maximum error of 1.0 percent compared to the analytical solution.

For elasticity, one analytical test problem consisted of a rectangular homogeneous plate under uniform tension at one end while having a fixed opposite boundary and zero tractions on the side walls. The finite element method solution of this direct problem was less than 1.0 percent in error. Another elasticity test case was also utilized where an annular pressure vessel was used to test the finite element method code (Martin et al., 1995). The finite element method solution of a direct problem was obtained when specifying tractions on both the inner and outer circular boundaries. The computed stress distributions were less than 2.0 percent in error compared to the analytical solution (Dennis and Dulikravich, 1998a).

Next, the combined thermoelastic analysis and inverse problems were attempted on an annular disk shown in Fig. 1. The outer circular boundary was constructed with 60 points while 30 points were used for the inner circular boundary. The triangular mesh contained 574 nodes and 242 quadratic elements. For the analysis problem, a temperature of 50.0 K was specified on the outer

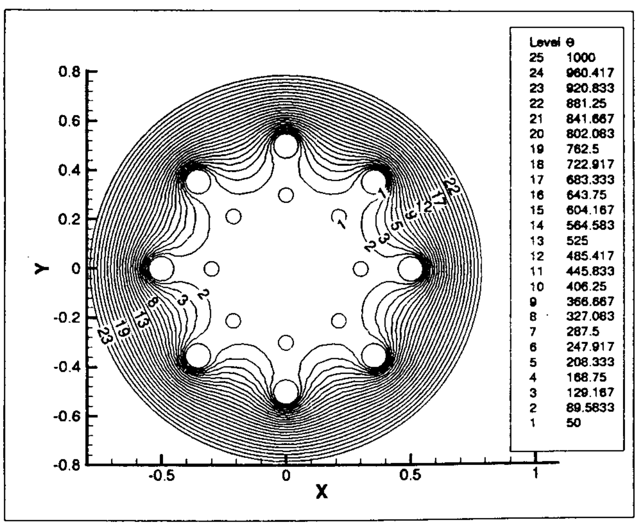

Fig. 9 Computed isotherms with inner and outer boundary temperatures specified

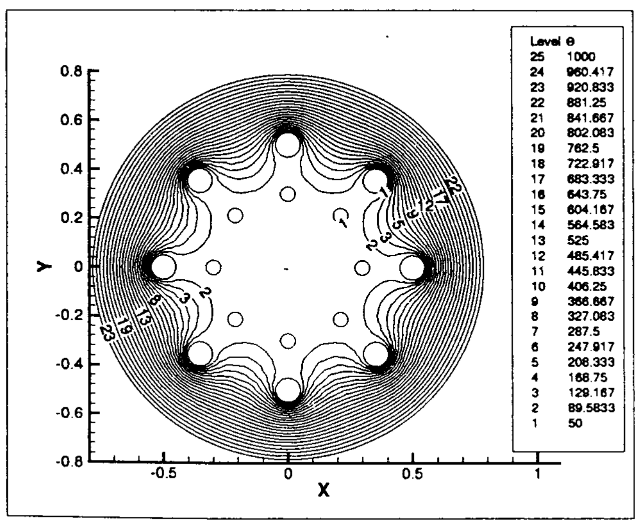

Fig. 10 Computed isotherms with outer boundary temperatures and fluxes specified. Nothing was specified on the inner boundary.

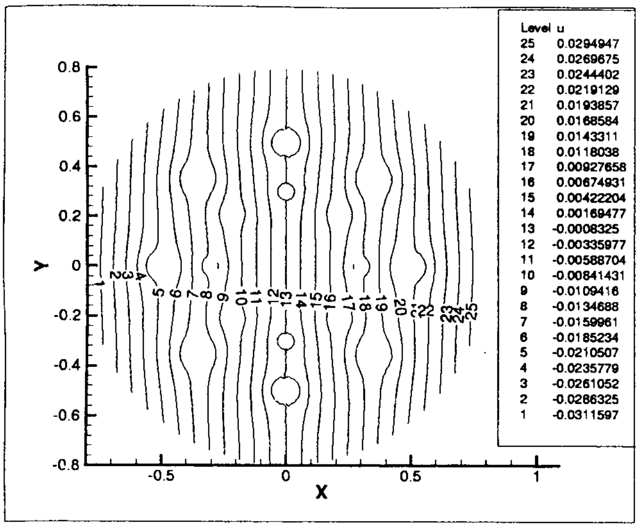

Fig. 11 Computed displacements in *x*-direction with inner and outer boundary tractions specified

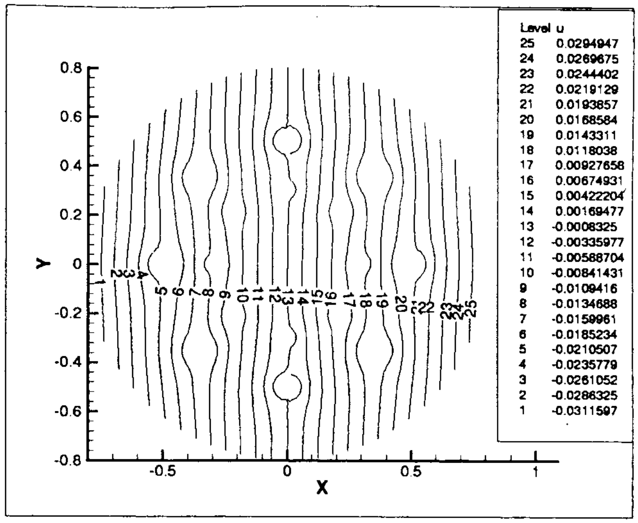

Fig. 12 Computed displacements in *x*-direction with outer boundary tractions and displacements specifies. Nothing was specified on the inner boundary.

circular boundary and 10.0 K was specified on the inner circular boundary. Simultaneously, a tensile surface stress of 101.0 kPa was specified on the outer circular boundary and a tensile stress of 202.0 kPa was specified on the inner circular boundary. The following material properties were used: $E = 2.0 \times 10^3$ Pa, $\nu = 2 \times 10^{-1}$, $\alpha = 2.0 \times 10^{-3}$ K^{-1}, and $k = 1.0$ Wm^{-1} K^{-1}. The computed temperature and stress distributions are shown in Figs. 3, 5, and 7.

The inverse problem was then created by overspecifying the outer circular boundary with the double-precision values of temperatures, fluxes, displacements, and tractions obtained from the numerical solution of the analysis problem. At the same time, no boundary conditions were specified on the inner circular boundary (Martin and Dulikravich, 1996a). A damping parameter of $\Lambda = 0$ was used. The computed temperature and stress distributions are shown in Figs. 4, 6, and 8. The maximum relative differences in temperatures, displacements, and stresses between the analysis and inverse results were less than 0.1 percent when solved with a QR factorization.

As a second thermoelastic test case, an analysis and an inverse problem were solved on the domain shown in Fig. 2. The domain was composed of 16 internal holes, each defined with 15 nodes. The outer circular boundary was constructed with 250 nodes. The triangular mesh contained 2310 nodes and 4170 linear elements.

For the analysis problem, a temperature of 1000.0 K was specified on the outer circular boundary and 50.0 K was specified on the 16 inner circular boundaries. A pressure of 101.0 kPa was applied to the outer boundary while a pressure of 202.0 kPa was applied to each of the 16 inner boundaries. The following material properties were used: $E = 2 \times 10^6$ Pa, $\nu = 10^{-1}$, $\alpha = 10^{-6}$ K^{-1}, $k = 1.0$ Wm^{-1} K^{-1}. The computed temperature and stress distributions from this well-posed (direct or analysis) problem are shown in Figs. 9, 11, and 13.

For the inverse problem, the boundary temperatures, fluxes, displacements, and tractions obtained from the forward analysis were specified on the outer circular boundary. No boundary conditions were specified on any of the 16 inner circular boundaries. Regularization method 3 was used. A damping parameter was $\Lambda = 1 \times 10^{-8}$ when determining the temperature field and $\Lambda = 1 \times 10^{-4}$ was used when computing the displacement field.

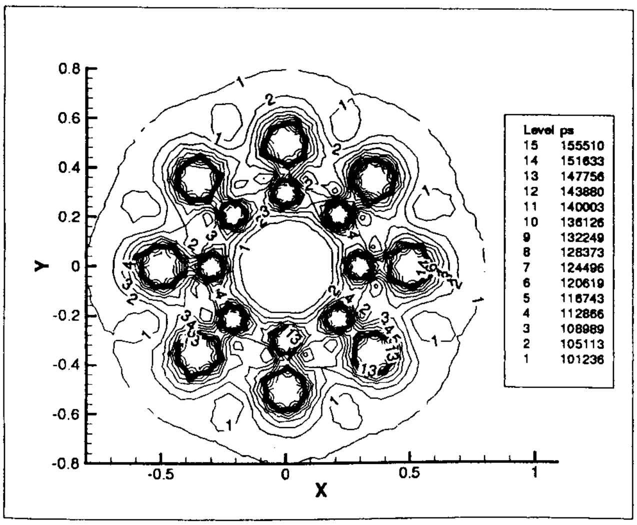

Fig. 13 Computed principal stress distribution with inner and outer boundary tractions specified

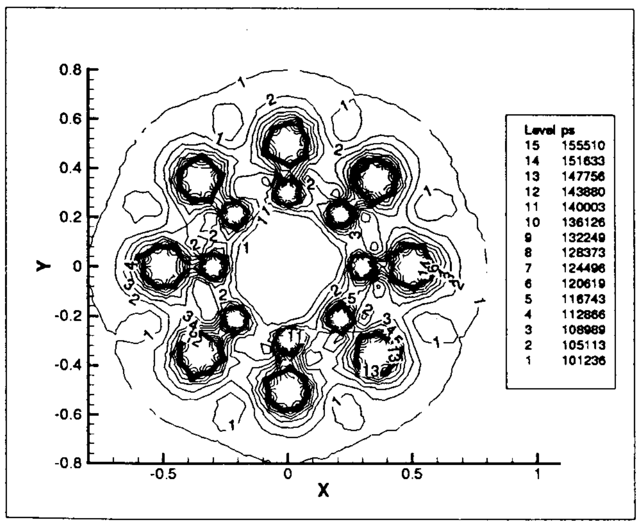

Fig. 14 Computed principal stress distribution with outer boundary tractions and displacements specified. Nothing was specified on the inner boundary.

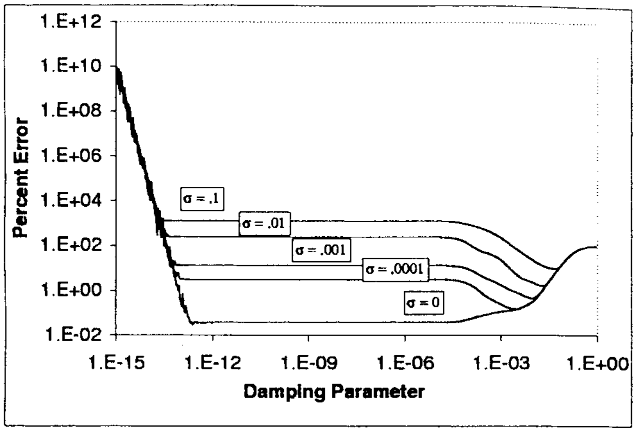

Fig. 15 Average error of predicted temperatures on unknown boundaries for regularization method 1 on annular region

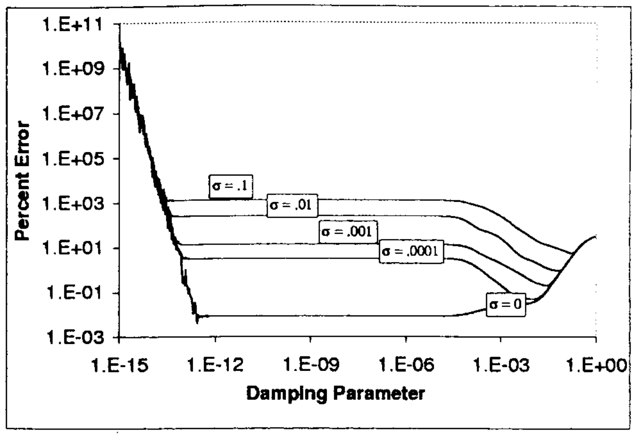

Fig. 16 Average error of predicted temperatures on unknown boundaries for regularization method 2 on annular region

The computed temperature and stress distributions from this ill-posed (inverse) problem are shown in Figs. 10, 12, and 14. The average relative differences between the numerical solutions of the forward and inverse temperatures, fluxes, displacements, and stresses were less than 0.1 percent when solved using a QR factorization.

Sensitivity to Input Error

The inverse heat conduction problem on the annular disk and multiply connected domain problems were tested with simulation of random measurement errors in the overspecified temperatures and fluxes. Random errors in the known boundary temperatures and fluxes were generated using the following equations (Martin and Dulikravich, 1996a):

$$\Theta = \Theta_{bc} \pm \sqrt{-2\bar{\sigma}^2 \ln R} \quad (41)$$

$$Q = Q_{bc} \pm \sqrt{-2\bar{\sigma}^2 \ln R} \quad (42)$$

For each case, Eqs. (41)–(42) were used to generate errors in both the known boundary temperatures and fluxes obtained from the numerical solution of the forward problem.

For the annular disk case, Figs. 15, 16, and 17 show the effect of the standard deviation, $\bar{\sigma}$, and damping parameter, Λ, on the average error of the temperatures recovered on the unknown boundaries compared to the temperatures given by the forward solution. Regularization method 3 gave the best results for all values of $\bar{\sigma}$. It produced errors in the unknown boundary conditions of the same magnitude as the input errors in the known boundary conditions.

For the multiply connected domain case, Figs. 18, 19, and 20 show the effect of the standard deviation, $\bar{\sigma}$, and damping parameter, Λ, on the average error of the temperatures recovered on the unspecified boundaries compared to the values given by the forward solution. None of the regularization methods worked well with this case when simulated measurement errors were applied. The input errors in the overspecified boundary conditions were amplified by several orders of magnitude in the temperatures predicted on the unspecified boundary. These results indicate that this finite element method inverse method requires better regularization if measurement errors are to be used with complicated multidomain geometries.

Discussion of Results

All three sparse matrix solvers performed well for test cases with relatively small number of variables. The QR factorization was found to provide the highest accuracy in the shortest computing time. For each of the test problems presented here the total solution time was less than five seconds on a Pentium 200 MHz PC. However, the QR factorization failed for larger problems where the number of grid points was greater than about 2000. This is most likely due to the instability of the QR

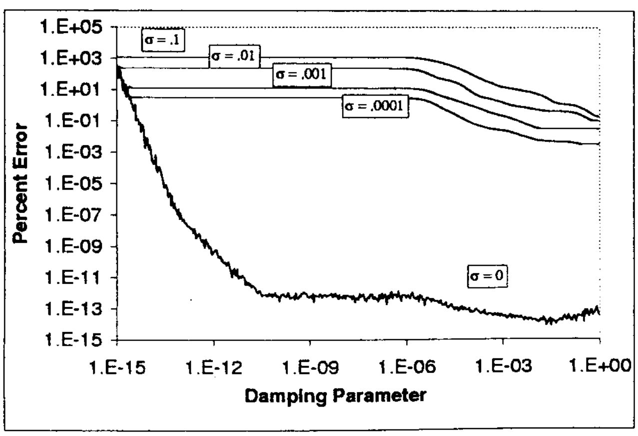

Fig. 17 Average error of predicted temperatures on unknown boundaries for regularization method 3 on annular region

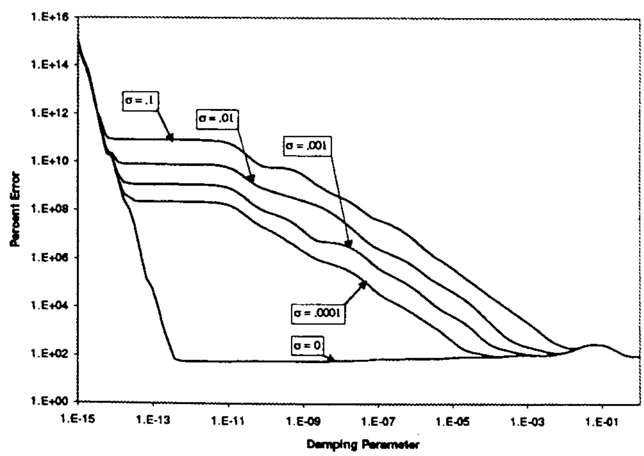

Fig. 18 Average error of predicted temperatures on unknown boundaries for regularization method 1 on multiply connected domain

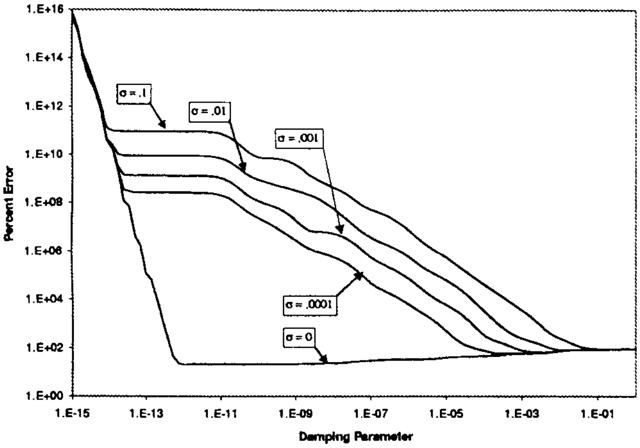

Fig. 19 Average error of predicted temperatures on unknown boundaries for regularization method 2 on multiply connected domain

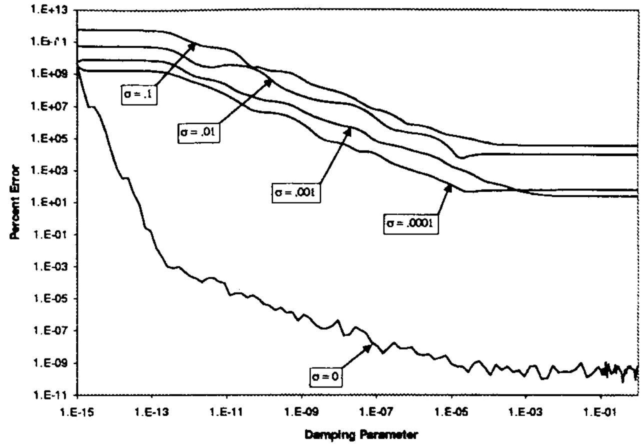

Fig. 20 Average error of predicted temperatures on unknown boundaries for regularization method 3 on multiply connected domain

algorithm when dealing with systems with high condition numbers (Golub and Van Loan, 1996). Applying small amounts of regularization ($\Lambda > 10^{-16}$) to the sparse matrix eliminated the instability. The CG method applied to the normalized equations worked well for problems with less than 100 nodes. For more than 100 nodes, this method required many iterations to converge to a solution less accurate than the QR solution. When regularization was applied to the sparse matrix, the CG convergence improved dramatically but the QR factorization was still much faster by comparison. The conjugate gradient least-square and least-squares QR methods were found to be slow for problems with more than 500 nodes, but were able to provide better solutions than those obtained with the CG applied to the normal equations.

Conclusion

A unified formulation for inverse determination of unknown steady boundary conditions in thermoelasticity has been developed and tested numerically using finite element method on several two-dimensional multiply connected configurations. The main conclusion is that the type and the amount of regularization used can significantly affect the accuracy of the results. This is true for the cases with no errors in the over-specified boundary conditions and for the cases with a random input error taken into account.

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