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UNIFIED ELECTRO-MAGNETO-FLUID DYNAMICS (EMFD): A SURVEY OF MATHEMATICAL MODELS

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Abstract—Fluid flow influenced by electric and magnetic fields has classically been divided into two separate, simplified categories: electro-hydrodynamics (EHD) studying flows under the influence of an electric field with free electric charges and no magnetic field, and magneto-hydrodynamics (MHD) studying flows under the influence of a magnetic field and no free electric charges or electric fields. This division was necessary to reduce the extreme complexity of the coupled system of Navier–Stokes, Maxwell’s and constitutive equations describing combined electro-magneto-hydrodynamic (EMFD) flows [G. S. Dulikravich and S. R. Lynn, Unified electro-magneto-fluid dynamics (EMFD): introductory concepts. *Int. J. Non-Linear Mechanics* 32, 913–922 (1997)]. In this paper, the unified EMFD theory is compared with classical EHD and MHD models. This reveals the inconsistencies and shortcomings of classical formulations and allows discussion of the relative importance of terms describing the electro-magnetic force, electric current and heat transfer.
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NOMENCLATURE

b	electric charge mobility coefficient, kg A s^2
\mathbf{B}	magnetic flux density vector, $\text{kg A}^{-1} \text{s}^{-2}$
$\underline{\underline{d}} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$	rate of deformation tensor, s^{-1}
\mathcal{D}	electric charge diffusion coefficient, $\text{m}^2 \text{s}^{-1}$
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	electric displacement field vector, A s m^{-2}
e	internal energy per unit mass, $\text{m}^2 \text{s}^{-2}$
$e_0 = e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{g} \cdot \mathbf{r}$	total energy per unit mass, $\text{m}^2 \text{s}^{-2}$
\mathbf{E}	electric field vector, $\text{kg m s}^{-3} \text{A}^{-1}$, or V m^{-1}
$\mathcal{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	electromotive intensity vector, $\text{kg m s}^{-3} \text{A}^{-1}$
\mathbf{f}	mechanical body force vector per unit mass, m s^{-2}
\mathbf{f}^{EM}	electromagnetic body force vector per unit volume, $\text{kg m}^{-2} \text{s}^{-2}$
\mathbf{g}	acceleration due to gravity, m s^{-2}
h	heat source or sink per unit mass, $\text{m}^2 \text{s}^{-3}$
$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$	magnetic field intensity vector, A m^{-1}
$\mathbf{J} = \mathbf{J}_c + \mathbf{J}_d$	electric current density vector, A m^{-2}
\mathbf{J}_c	electric conduction current vector, A m^{-2}
\mathbf{J}_d	electric drift current vector, A m^{-2}
\mathbf{M}	total magnetization vector per unit volume, A m^{-1}
$\mathcal{M} = \mathbf{M} + \mathbf{v} \times \mathbf{P}$	magnetomotive intensity vector per unit volume, A m^{-1}
p	pressure, $\text{kg m}^{-1} \text{s}^{-2}$
\mathbf{P}	total polarization vector per unit volume, A s m^{-2}
q_e	local free electric charge per unit volume, A s m^{-3}
q_o	total or free electric charge per unit volume, A s m^{-3}
\mathbf{q}	heat flux vector, kg s^{-3}
\mathbf{r}	position vector, m
s	entropy per unit mass, $\text{m}^2 \text{kg}^{-1} \text{K}^{-1} \text{s}^{-2}$

$\underline{\mathbf{t}} = -p\underline{\mathbf{I}} + \underline{\boldsymbol{\tau}}^v + \underline{\boldsymbol{\tau}}^{\text{EM}}$	fluid stress tensor, $\text{kg m}^{-1} \text{s}^{-2}$
$\underline{\mathbf{v}}$	fluid velocity vector, m s^{-1}
\mathcal{V}	volume, m^3
<i>Greek symbols</i>	
ϵ	dielectric constant or electric permittivity, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_0 = 8.854 \times 10^{-12}$	vacuum dielectric constant or electric permittivity, $\text{kg}^{-1} \text{m}^{-3} \text{s}^4 \text{A}^2$
$\epsilon_r = \epsilon/\epsilon_0$	relative electric permittivity
κ	thermal conductivity coefficient, $\text{kg m s}^{-3} \text{K}^{-1}$
λ_v	second coefficient of viscosity, $\text{kg m}^{-1} \text{s}^{-1}$
σ	electric conductivity coefficient, $\text{kg}^{-1} \text{m}^{-3} \text{s}^3 \text{A}^2$
θ	absolute temperature, K
ρ	fluid density, kg m^{-3}
$\underline{\boldsymbol{\tau}}^v$	viscous stress tensor, $\text{kg m}^{-1} \text{s}^{-2}$
$\underline{\boldsymbol{\tau}}^{\text{EM}}$	electromagnetic stress tensor, $\text{kg m}^{-1} \text{s}^{-2}$
μ	magnetic permeability coefficient, $\text{kg m A}^{-2} \text{s}^{-2}$
$\mu_0 = 4\pi \times 10^{-7}$	magnetic permeability of vacuum, $\text{kg m A}^{-2} \text{s}^{-2}$
$\mu_r = \mu/\mu_0$	relative magnetic permeability
μ_v	shear coefficient of viscosity, $\text{kg m}^{-1} \text{s}^{-1}$
$\chi^E = \epsilon_r - 1$	electric susceptibility
$\chi^M = \mu_r - 1$	magnetic susceptibility
$\Phi = \underline{\boldsymbol{\tau}}^v : \underline{\mathbf{d}}$	viscous dissipation function, $\text{kg m}^{-1} \text{s}^{-3}$
$\Psi = e - \theta s$	material free energy function, $\text{m}^2 \text{s}^{-2}$

1. INTRODUCTION

The equations governing electro-magneto-fluid dynamic (EMFD) flows consist of the Navier–Stokes equations of fluid motion coupled with Maxwell’s equations of electro-magnetics and material constitutive relations. The field has traditionally been divided into flows influenced only by electric fields and electric charges, and flows influenced only by magnetic fields and without electric charges. The former are called Electro-Hydrodynamic (EHD) flows and the latter Magneto-Hydrodynamic (MHD) flows [2]. Studies of EHD and MHD flows have ranged in complexity from the experimentally-based [3] to more theoretically-based [4]. Much more recently, rigorous theoretical continuum mechanics treatments of EHD [5] and unified EMFD flows [6, 7] have been developed. These continuum mechanics approaches are limited to non-relativistic, relatively low frequency phenomena [8, 9] up to approx. 10^3 Hz.

Part 1 [1] presented an overview of electro-magnetic theory with concentrated effort placed on the field–material interactions of polarization and magnetization. The unified EMFD theory [6, 7] was also succinctly presented in Part 1. Presented in Part 2 is an overview of classical EHD and MHD models. The mainstay of this paper, however, is a comparison between classical EHD and MHD models and the unified EMFD theory. The comparison concentrates on similarities and differences between electro-magnetic force, electric current and heat conduction terms in the classical and unified models. Included in this is a discussion of the physical meaning and relative importance of classical model terms and recommendations for improving classical models. The inadequacy of simple super-positioning of classical models to fully describe unified EMFD flows is also noted.

2. GOVERNING SYSTEM OF EQUATIONS

The full system of equations governing unified EMFD flow consists of the Maxwell’s equations governing electro-magnetism, the Navier–Stokes equations governing fluid flow and constitutive equations describing material behavior. Maxwell’s equations are the system of linear differential equations governing electro-magnetic fields. They are given as [7, p. 504]

Gauss' law

$$\nabla \cdot \underline{\mathbf{D}} = q_o \quad (1)$$

Ampere–Maxwell's law

$$\frac{\partial \underline{\mathbf{D}}}{\partial t} - \nabla \times \underline{\mathbf{H}} = - \underline{\mathbf{J}} \quad (2)$$

or

$$\frac{\partial \underline{\mathbf{D}}}{\partial t} - \nabla \times \frac{\underline{\mathbf{B}}}{\mu_0} = - \underline{\mathbf{J}} - \nabla \times \underline{\mathbf{M}} \quad (3)$$

Conservation of magnetic flux

$$\nabla \cdot \underline{\mathbf{B}} = 0 \quad (4)$$

Faraday's law

$$\frac{\partial \underline{\mathbf{B}}}{\partial t} + \nabla \times \underline{\mathbf{E}} = 0 \quad (5)$$

Conservation of electric charges

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \quad (6)$$

Detailed descriptions of these equations can be found in any number of texts [10–12]. The equations of motion governing EMFD flow are the Navier–Stokes relations into which electromagnetic effects have been included. A rigorous derivation of these equations for electro-magnetic fluids is completed by Eringen and Maugin [6, p. 129].

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\mathbf{v}}) = 0 \quad (7)$$

Conservation of momentum

$$\frac{\partial (\rho \underline{\mathbf{v}})}{\partial t} + \nabla \cdot (\underline{\mathbf{v}} \rho \underline{\mathbf{v}} - \underline{\mathbf{t}}) - \rho \underline{\mathbf{f}} - \underline{\mathbf{f}}^{\text{EM}} = 0 \quad (8)$$

where the electromagnetic force per unit volume is [6, p. 59]

$$\underline{\mathbf{f}}^{\text{EM}} = q_o \underline{\mathbf{E}} + \underline{\mathbf{J}} \times \underline{\mathbf{B}} + (\nabla \underline{\mathbf{E}}) \cdot \underline{\mathbf{P}} + (\nabla \underline{\mathbf{B}}) \cdot \underline{\mathbf{M}} + \nabla \cdot (\underline{\mathbf{v}} \underline{\mathbf{P}} \times \underline{\mathbf{B}}) + \frac{\partial}{\partial t} (\underline{\mathbf{P}} \times \underline{\mathbf{B}}) \quad (9)$$

Conservation of energy

$$\frac{\partial (\rho e_o)}{\partial t} + \nabla \cdot (\rho e_o \underline{\mathbf{v}}) - \nabla \cdot (\underline{\mathbf{t}} \cdot \underline{\mathbf{v}}) + \nabla \cdot \underline{\mathbf{q}} - \rho h - \rho \underline{\mathcal{E}} \cdot \frac{D \left(\frac{\underline{\mathbf{P}}}{\rho} \right)}{Dt} + \underline{\mathcal{M}} \cdot \frac{D \underline{\mathbf{B}}}{Dt} - \underline{\mathbf{J}}_c \cdot \underline{\mathcal{E}} = 0 \quad (10)$$

The fluid (material) constitutive equations complete the system of equations governing EMFD flows and are given below. A rigorous derivation of these relations is given by Eringen and Maugin [6, 7]. As in Part 1 of this paper, it must be noted that the electromagnetic material properties of the fluid may be dependent on physical properties of the flow, especially electro-magnetic frequency and temperature. With this in mind, the unified EMHD free energy, polarization, magnetization, viscous and electromagnetic stress, thermodynamic pressure, electric conduction current and heat flux constitutive equations for non-linear, relatively low frequency ($< 10^3$ Hz) materials may be given as [6, pp. 177–178]

$$\Psi = \Psi(I_1, I_2, I_3, \theta, \rho^{-1}) \quad (11)$$

$$\underline{\mathbf{P}} = - 2\rho \left(\frac{\partial \Psi}{\partial I_1} \underline{\mathcal{E}} + \frac{\partial \Psi}{\partial I_3} (\underline{\mathcal{E}} \cdot \underline{\mathbf{B}}) \underline{\mathbf{B}} \right) \quad (12)$$

$$\underline{m} = -2\rho \left(\frac{\partial \Psi}{\partial I_2} \underline{\mathbf{B}} + \frac{\partial \Psi}{\partial I_3} (\underline{\mathcal{E}} \cdot \underline{\mathbf{B}} \underline{\mathbf{B}}) \right) \quad (13)$$

$$\underline{\mathbf{t}} = -p \underline{\mathbf{I}} + \underline{\tau}(\rho^{-1}, \underline{\mathbf{d}}, \underline{\mathcal{E}}, \underline{\mathbf{B}}, \theta, \nabla \theta) \quad (14)$$

$$p = -\frac{\partial \Psi}{\partial \rho^{-1}} \quad (15)$$

$$\begin{aligned} \underline{\mathbf{J}}_c = & (\sigma_1 \underline{\mathcal{E}} + \sigma_2 \underline{\mathbf{d}} \cdot \underline{\mathcal{E}} + \sigma_3 \underline{\mathbf{d}}^2 \cdot \underline{\mathcal{E}}) + (\sigma_4 \nabla \theta + \sigma_5 \underline{\mathbf{d}} \cdot \nabla \theta + \sigma_6 \underline{\mathbf{d}}^2 \cdot \nabla \theta) \\ & + (\sigma_7 \underline{\mathcal{E}} \times \underline{\mathbf{B}} + \sigma_8 (\underline{\mathbf{d}} \cdot (\underline{\mathcal{E}} \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \underline{\mathcal{E}}) \times \underline{\mathbf{B}})) \\ & + (\sigma_9 \nabla \theta \times \underline{\mathbf{B}} + \sigma_{10} (\underline{\mathbf{d}} \cdot (\nabla \theta \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \nabla \theta) \times \underline{\mathbf{B}})) \\ & + \sigma_{11} (\underline{\mathbf{B}} \cdot \underline{\mathcal{E}}) \underline{\mathbf{B}} + \sigma_{12} (\underline{\mathbf{B}} \cdot \nabla \theta) \underline{\mathbf{B}} \end{aligned} \quad (16)$$

$$\begin{aligned} \underline{\mathbf{q}} = & (\kappa_1 \underline{\mathcal{E}} + \kappa_2 \underline{\mathbf{d}} \cdot \underline{\mathcal{E}} + \kappa_3 \underline{\mathbf{d}}^2 \cdot \underline{\mathcal{E}}) + (\kappa_4 \nabla \theta + \kappa_5 \underline{\mathbf{d}} \cdot \nabla \theta + \kappa_6 \underline{\mathbf{d}}^2 \cdot \nabla \theta) \\ & + (\kappa_7 \underline{\mathcal{E}} \times \underline{\mathbf{B}} + \kappa_8 (\underline{\mathbf{d}} \cdot (\underline{\mathcal{E}} \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \underline{\mathcal{E}}) \times \underline{\mathbf{B}})) \\ & + (\kappa_9 \nabla \theta \times \underline{\mathbf{B}} + \kappa_{10} (\underline{\mathbf{d}} \cdot (\nabla \theta \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \nabla \theta) \times \underline{\mathbf{B}})) \\ & + \kappa_{11} (\underline{\mathbf{B}} \cdot \underline{\mathcal{E}}) \underline{\mathbf{B}} + \kappa_{12} (\underline{\mathbf{B}} \cdot \nabla \theta) \underline{\mathbf{B}} \end{aligned} \quad (17)$$

where

$$\underline{\mathcal{E}} = \underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}} \quad (18)$$

$$I_1 = \underline{\mathcal{E}} \cdot \underline{\mathcal{E}} \quad (19)$$

$$I_2 = \underline{\mathbf{B}} \cdot \underline{\mathbf{B}} \quad (20)$$

$$I_3 = (\underline{\mathcal{E}} \cdot \underline{\mathbf{B}})^2 \quad (21)$$

$$\begin{aligned} \underline{\tau} = & \alpha_0 \underline{\mathbf{I}} + \alpha_1 \underline{\mathbf{d}} + \alpha_2 \underline{\mathbf{d}}^2 + \alpha_3 \underline{\mathcal{E}} \otimes \underline{\mathcal{E}} + \alpha_4 \underline{\mathbf{B}} \otimes \underline{\mathbf{B}} + \alpha_5 \nabla \theta \otimes \nabla \theta \\ & + \alpha_6 (\underline{\mathcal{E}} \otimes \underline{\mathbf{d}} \cdot \underline{\mathcal{E}})_s + \alpha_7 (\underline{\mathcal{E}} \otimes \underline{\mathbf{d}}^2 \cdot \underline{\mathcal{E}})_s + \alpha_8 (\nabla \theta \otimes \underline{\mathbf{d}} \cdot \nabla \theta)_s \\ & + \alpha_9 (\nabla \theta \otimes \underline{\mathbf{d}}^2 \cdot \nabla \theta)_s + \alpha_{10} (\underline{\mathbf{d}} \cdot \underline{\mathbf{W}} - \underline{\mathbf{W}} \cdot \underline{\mathbf{d}}) + \alpha_{11} \underline{\mathbf{W}} \cdot \underline{\mathbf{d}} \cdot \underline{\mathbf{W}} \\ & + \alpha_{12} (\underline{\mathbf{d}}^2 \cdot \underline{\mathbf{W}} - \underline{\mathbf{W}} \cdot \underline{\mathbf{d}}^2) + \alpha_{13} (\underline{\mathbf{W}} \cdot \underline{\mathbf{d}} \cdot \underline{\mathbf{W}}^2 - \underline{\mathbf{W}}^2 \cdot \underline{\mathbf{d}} \cdot \underline{\mathbf{W}}) \\ & + \alpha_{14} (\underline{\mathcal{E}} \otimes \nabla \theta)_s + \alpha_{15} (\underline{\mathbf{W}} \cdot \underline{\mathcal{E}} \otimes \underline{\mathcal{E}} \cdot \underline{\mathbf{W}})_s + \alpha_{16} (\underline{\mathcal{E}} \otimes \underline{\mathbf{W}} \cdot \underline{\mathcal{E}})_s \\ & + \alpha_{17} (\underline{\mathbf{W}} \cdot \underline{\mathcal{E}} \otimes \underline{\mathbf{W}}^2 \cdot \underline{\mathcal{E}})_s + \alpha_{18} \underline{\mathbf{d}} \cdot (\underline{\mathcal{E}} \otimes \nabla \theta - \nabla \theta \otimes \underline{\mathcal{E}}) \\ & + \alpha_{18} (\underline{\mathcal{E}} \otimes \nabla \theta - \nabla \theta \otimes \underline{\mathcal{E}}) \cdot \underline{\mathbf{d}} + \alpha_{19} \underline{\mathbf{W}} \cdot (\underline{\mathcal{E}} \otimes \nabla \theta - \nabla \theta \otimes \underline{\mathcal{E}})_s \end{aligned} \quad (22)$$

$$\underline{\mathbf{W}} = W_{ij} = \varepsilon_{ijk} B_k \quad (23)$$

In the above relations the tensor $\underline{\mathbf{d}}^2$ is the product of the rate of deformation tensor $\underline{\mathbf{d}}^2 = \underline{\mathbf{d}} \cdot \underline{\mathbf{d}}$. Additionally, note that the material coefficients σ_i , κ_i , and α_i are functions of the joint invariants $\underline{\mathbf{d}}$, $\underline{\mathcal{E}}$, $\underline{\mathbf{B}}$ and $\nabla \theta$ [6, p. 13, 178]. Finally, the subscript s attached to brackets in equation (22) indicates symmetrization.

Most classical models consider only isotropic materials. This assumption will be considered here as well. Under the isotropic assumption, all tensor quantities in the electric conduction current and heat conduction relations and most in the stress relation become zero. The reason for removal of cross-product and tensor quantities in the linear, isotropic constitutive relations is that the order of the integrity basis for the governing constitutive relations has been lowered [6, pp. 154, 173]. This leads to the linear constitutive relations for relatively low frequency materials and may be given as

$$\Psi = \Psi_0 - \frac{1}{2\rho} \left(\varepsilon_0 \chi^E I_1 + \frac{\chi^M}{\mu_0 (1 + \chi^M)} I_2 \right) \quad (24)$$

$$\underline{\mathbf{P}} = -2\rho \frac{\partial \Psi}{\partial I_1} \underline{\mathcal{E}} = \varepsilon_0 \chi^E \underline{\mathcal{E}} \quad (25)$$

$$\underline{m} = -2\rho \frac{\partial \Psi}{\partial I_2} \underline{\mathbf{B}} = \frac{\chi^M}{\mu_0 (1 + \chi^M)} \underline{\mathbf{B}} \quad (26)$$

$$\underline{\mathbf{t}} = -p\underline{\mathbf{I}} + \underline{\chi}(\rho^{-1}, \underline{\mathbf{d}}, \theta) = -p\underline{\mathbf{I}} + 2\mu_v \underline{\mathbf{d}} + \lambda_v \text{tr}(\underline{\mathbf{d}})\underline{\mathbf{I}} \quad (27)$$

$$p = -\frac{\partial \Psi}{\partial \rho^{-1}} \quad (28)$$

$$\underline{\mathbf{J}}_c = \sigma \underline{\mathcal{E}} + \sigma^\theta \nabla \theta \quad (29)$$

$$\underline{\mathbf{q}} = \kappa^{\text{EM}} \underline{\mathcal{E}} + \kappa \nabla \theta \quad (30)$$

where the coefficients χ^E , χ^M , μ_v , λ_v , σ , σ^θ , κ^{EM} and κ are functions of ρ and θ .

It can be seen from equations (16) and (17) that the electromagnetic field is not the only cause of electric current and the temperature gradient is not the only source of heat conduction as is commonly assumed. The electric field, magnetic field and heat conduction may couple to produce electric charge motion and heat transfer. These couplings are called phenomenological cross effects and may be placed in four general categories: (1) thermoelectric, (2) galvanomagnetic, (3) thermomagnetic, and (4) second order effects [6, pp. 161–163]. These categories are based on the source of the effect and each will be described separately.

2.1. Thermoelectric effects

Thermoelectric effects are caused by couplings between the temperature gradient and the electric field. Reducing the current and heat conduction equations to their thermoelectric terms yields

$$\underline{\mathbf{J}}_c = (\sigma_4 \nabla \theta + \sigma_5 \underline{\mathbf{d}} \cdot \nabla \theta + \sigma_6 \underline{\mathbf{d}}^2 \cdot \nabla \theta) \quad (31)$$

$$\underline{\mathbf{q}} = (\kappa_1 \underline{\mathcal{E}} + \kappa_2 \underline{\mathbf{d}} \cdot \underline{\mathcal{E}} + \kappa_3 \underline{\mathbf{d}}^2 \cdot \underline{\mathcal{E}}) \quad (32)$$

A temperature gradient producing an electric current is referred to as the Thomson effect while an electric field producing heat conduction is termed the Peltier effect. These two effects together are known as the Seebeck effect and form the basis for thermocouples. Also note that the first term in equation (16) and the fourth term in equation (17) are not true cross effects; they are the Ohmic charge conduction and Fourier heat transfer, respectively.

2.2. Galvanomagnetic effects

When the electric and magnetic fields are not parallel, electric current and heat conduction are induced in the material. These sets of effects are termed galvanomagnetic. Reducing the full current and heat conduction equations to their galvanomagnetic effects yields

$$\underline{\mathbf{J}}_c = (\sigma_7 \underline{\mathcal{E}} \times \underline{\mathbf{B}} + \sigma_8 (\underline{\mathbf{d}} \cdot (\underline{\mathcal{E}} \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \underline{\mathcal{E}}) \times \underline{\mathbf{B}})) \quad (33)$$

$$\underline{\mathbf{q}} = (\kappa_7 \underline{\mathcal{E}} \times \underline{\mathbf{B}} + \kappa_8 (\underline{\mathbf{d}} \cdot (\underline{\mathcal{E}} \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \underline{\mathcal{E}}) \times \underline{\mathbf{B}})) \quad (34)$$

Electric current induction from non-parallel electric and magnetic fields is called the Hall effect. In analogy, heat conduction produced by non-parallel electric and magnetic fields is called the Ettingshausen effect [6, pp. 161–163].

2.3. Thermomagnetic effects

When the temperature gradient and the magnetic field are not parallel, electric current (Nernst effect) and heat conduction (Righi–LeDuc effect) are induced in the material. These effects are termed thermomagnetic. Reducing the full conduction current and heat conduction equations to their thermomagnetic effects yields

$$\underline{\mathbf{J}}_c = (\sigma_9 \nabla \theta \times \underline{\mathbf{B}} + \sigma_{10} (\underline{\mathbf{d}} \cdot (\nabla \theta \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \nabla \theta) \times \underline{\mathbf{B}})) \quad (35)$$

$$\underline{\mathbf{q}} = (\kappa_9 \nabla \theta \times \underline{\mathbf{B}} + \kappa_{10} (\underline{\mathbf{d}} \cdot (\nabla \theta \times \underline{\mathbf{B}}) - (\underline{\mathbf{d}} \cdot \nabla \theta) \times \underline{\mathbf{B}})) \quad (36)$$

2.4. Secondary effects

Finally, secondary effects are caused by non-orthogonality between the electric field and magnetic field and the temperature gradient and magnetic field. The electric current and heat conduction equations with only secondary effects are shown below.

$$\underline{\mathbf{J}}_c = \sigma_{11}(\underline{\mathbf{B}} \cdot \underline{\mathcal{E}})\underline{\mathbf{B}} + \sigma_{12}(\underline{\mathbf{B}} \cdot \nabla\theta)\underline{\mathbf{B}} \quad (37)$$

$$\underline{\mathbf{q}} = \kappa_{11}(\underline{\mathbf{B}} \cdot \underline{\mathcal{E}})\underline{\mathbf{B}} + \kappa_{12}(\underline{\mathbf{B}} \cdot \nabla\theta)\underline{\mathbf{B}} \quad (38)$$

3. CLASSICAL ELECTRO-HYDRODYNAMICS

As mentioned previously EHD flows are those in which magnetic effects may be neglected and charged particles are present. One of the implied assumptions is that the flows are at non-relativistic speeds. There are cases, particularly in astrophysical MHD, where this assumption cannot be made [12]. The other major assumption made in classical EHD is that only a quasi-static electric field is applied so that the magnetic field, both applied and induced, may be neglected. Atten and Moreau [13] present a detailed coverage of classical EHD modeling and discuss the relative importance of terms in the force and electric current through stability analysis. In order for this assumption to apply the fluid must contain electrically charged particles. With these assumptions, three Maxwell's equations govern the flow [2].

$$\nabla \cdot \underline{\mathbf{D}} = q_0 \quad (39)$$

$$\nabla \times \underline{\mathbf{E}} = 0 \quad (40)$$

$$\frac{\partial q_0}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \quad (41)$$

Modifications to the Navier–Stokes relations come from the electro-magnetic force on the fluid from which all magnetic field terms have been neglected. With classical EHD assumptions the electro-magnetic force in the unified EMFD theory in equation (9) becomes

$$\underline{\mathbf{f}}^{\text{EM}} = q_0 \underline{\mathbf{E}} + (\nabla \underline{\mathbf{E}}) \cdot \underline{\mathbf{P}} \quad (42)$$

This is not the form of the electro-magnetic force usually seen in classical EHD formulations [1]. Through the use of thermodynamics, equation (42) and the material constitutive equation of state, the electro-magnetic force is usually manipulated into the following equivalent forms [4, pp. 59–63; 6, pp. 505–507]

$$\underline{\mathbf{f}}^{\text{EM}} = q_0 \underline{\mathbf{E}} - \frac{\underline{\mathbf{E}}^2}{2} \nabla \varepsilon + \frac{1}{2} \nabla \left(\underline{\mathbf{E}}^2 \rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{\theta} \right) \quad (43)$$

or as

$$\underline{\mathbf{f}}^{\text{EM}} = q_0 \underline{\mathbf{E}} - \frac{\underline{\mathbf{E}}^2}{2} \left(\frac{\partial \varepsilon}{\partial \theta} \right)_{\rho} \nabla \theta + \frac{\rho}{2} \nabla \left(\underline{\mathbf{E}}^2 \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{\theta} \right) \quad (44)$$

Equation (44) is the most common electro-magnetic force formulation in classical EHD. The three terms in the equation are the electrophoretic, dielectrophoretic and electrostrictive terms, respectively. The *electrophoretic force* or Coulomb force is caused by the electric field acting on free charges in the fluid. It is an irrotational force except when charge gradients are present [14]. The *dielectrophoretic force* is also a translational force, but is caused by polarization of the fluid and/or particles in the fluid. A dielectrophoretic force will occur where high gradients of electric permittivity are present. This condition will be true in high temperature gradient flows, multi-constituent flows, particulate flows [15] or any time the electric field must pass through two contacting media of different permittivities [16]. Grassi and DiMarco [17] treat the dielectrophoretic force as it applies to bubbly flows and heat transfer. Poulter and Allen [14] note that the dielectrophoretic force produces greatest circulation when the dielectric permittivity is inhomogeneous and non-parallel with the applied electric field. The last force, the *electrostrictive force*, is a distortive force (as opposed

to the previous translational forces) associated with fluid compression and shear. The electrostrictive force is usually smaller than the -phoretic forces, but is present in high pressure gradient flows, compressible flows and flows with a non-uniform applied electric field. In hydrodynamically bounded systems, the electrostrictive force plays no part due to its irrotational nature [14]. Pohl [15] describes this phenomenon in greater detail.

Classical EHD modeling derives directly from the unified EMFD theory. The same can be said of the constitutive current relation. From the unified EMFD theory the electric current, assuming material isotropy and linearity, is given by the relation

$$\underline{\mathbf{J}} = q_0 \underline{\mathbf{v}} + \sigma \underline{\mathcal{E}} + \sigma^{\theta} \nabla \theta \tag{45}$$

This is not the form seen in classical EHD models however [2]. Classical EHD modeling typically defines the electric current as simply the first two terms of equation (45) : the convective and conductive current, respectively. However, more advanced classical models define the electric current as [6, p. 562]

$$\underline{\mathbf{J}} = q_0 \underline{\mathbf{v}} + \sigma \underline{\mathbf{E}} + q_0 b \underline{\mathbf{E}} - \mathcal{D} \nabla q_0 \tag{46}$$

At first glance equation (45) seems not to match equation (46), or, seems to imply that the temperature gradient is directly related to the electric charge gradient. This may be shown to be true as the last two terms in equation (46) come from the Einstein–Fokker relationships, derived from studies of Brownian motion [18, p. 264–273], which relate any concentration gradient to a mobility ($q_0 b \underline{\mathbf{E}}$) and a diffusion ($\mathcal{D} \nabla q_0$). A more heuristic proof of this relation can be obtained by expanding the material equation of state, which by linear theory relates pressure, temperature and density, in a Taylor series around density and pressure. Then, because the electric current is desired, we consider only electric field and charge contributions to the pressure. Thus, the gradient of velocity becomes an equivalent electric charge gradient and the gradient of pressure becomes an equivalent of the electrophoretic force (both multiplied by constants). By performing a unit analysis on the constants, it can be seen that the constant multiplied by the charge gradient is equal to the charge mobility coefficient, b , and the constant multiplied by the charge gradient is equal to the charge diffusion coefficient, \mathcal{D} . By either de Groot and Mazur’s rigorous non-equilibrium thermodynamics method or method described above, equation (45) may be shown to be equivalent to equation (46). Newman [19] also provides a detailed discussion of the concepts of diffusion and mobility. The second, or diffusive term, is often neglected where limited free charges are available [20].

The final equation describing classical EHD flow is the heat transfer constitutive relation. This is the one area where classical EHD theory does not match the unified EMFD theory with EHD assumptions. By introducing classical EHD assumptions in the unified EMFD theory the following relationship is obtained for heat transfer

$$\underline{\mathbf{q}} = \kappa_1 \underline{\mathbf{E}} + \kappa_4 \nabla \theta \tag{47}$$

This relationship, however is not the one commonly seen in classical EHD models. These models usually neglect the contribution of heat transfer from the electric field so that equation (47) becomes

$$\underline{\mathbf{q}} = \kappa_4 \nabla \theta \tag{48}$$

This relation is Fourier’s law of heat conduction. Here, an inconsistency must be noted that although classical EHD modeling seems to neglects heat transfer induced by the electric field and electric current, Joule heating $\underline{\mathbf{J}}_c \cdot \underline{\mathcal{E}}$ from equation (10), is usually included in the EHD analysis.

4. CLASSICAL MAGNETO-HYDRODYNAMICS

The classical modeling of magneto-hydrodynamics makes the non-relativistic assumption just as in classical EHD theory. However, where classical EHD made the assumption of quasi-electrostatics, classical MHD theory makes a quasi-magnetostatic assumption. Thus, the direct electric field induces a magnetic field of much less magnitude than the applied magnetic field. This assumption also implies that electric current comes primarily from

conductive means and that there are no free charges in the fluid. With these assumptions Maxwell's equations become [2]

$$\nabla \cdot \mathbf{B} = 0 \quad (49)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (50)$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (51)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (52)$$

Another, perhaps more commonly used, MHD assumption is that the material is a perfect conductor ($\sigma = \infty$) [7, p. 512]. This assumption makes the Joule heating $\mathbf{J}_c \cdot \mathcal{E}$ in equation (10) infinite as well. For Joule heating to remain finite requires the assumption that the electromotive intensity vector is zero. This in turn gives a relation for the electric field in terms of the magnetic field and velocity vector and may be shown as

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \quad (53)$$

Substitution into equation (50) and into the Navier–Stokes equations allows the MHD equations to be free of any applied or induced electric field.

The modifications to the Navier–Stokes relations come from the electro-magnetic force on the fluid from which all induced electric field terms have been neglected. Otherwise, the equations appear the same as in Section 2. With classical MHD assumptions the electro-magnetic force in the unified EMFD theory becomes [2]

$$\mathbf{f}^{\text{EM}} = \mathbf{J} \times \mathbf{B} + (\nabla \mathbf{B}) \cdot \mathbf{M} \quad (54)$$

The second term, source of dimagnetophoretic and magnetostrictive forces is typically neglected in MHD. Thus, the electro-magnetic force for classical MHD becomes [2]

$$\mathbf{f}^{\text{EM}} = \mathbf{J} \times \mathbf{B} \quad (55)$$

An improvement in classical MHD modeling could be made by including the dimagnetophoretic and magnetostrictive terms, especially in cases where MHD flow conditions satisfy conditions analogous to cases where dielectrophoretic and electrostrictive effects are important in EHD. By making classical MHD assumptions, the unified EMFD current becomes

$$\mathbf{J} = \sigma_1 \mathcal{E} + \sigma_4 \nabla \theta + \sigma_7 \mathcal{E} \times \mathbf{B} + \sigma_9 \nabla \theta \times \mathbf{B} + \sigma_{11} (\mathbf{B} \cdot \mathcal{E}) \mathbf{B} + \sigma_{12} (\mathbf{B} \cdot \nabla \theta) \mathbf{B} \quad (56)$$

However, classical MHD theory usually defines the current as [7, p. 510]

$$\mathbf{J} = \sigma \mathbf{E} + \sigma (\mathbf{v} \times \mathbf{B}) + \sigma^\theta \nabla \theta = \sigma \mathcal{E} + \sigma^\theta \nabla \theta \quad (57)$$

Here, σ^θ is the Seebeck coefficient [6, p.174]. It is seen to be equivalent to σ_4 in equation (56). Note that in some classical MHD formulations the Seebeck coefficient is not used [2]. Regardless, the classical MHD formulation neglects a significant number of effects. Improvements could be made to the classical MHD theory by including terms from equation (56) depending on the details of the flow problem in question.

The final relation of comparison between the unified EMFD model and the classical MHD model is the heat transfer. Once again in classical modeling, Joule heating is often included in the energy relation, but the heat transfer constitutive relation remains the same as in equation (48). In comparison the unified EMFD model with classical MHD assumptions is

$$\mathbf{q} = \kappa_1 \mathcal{E} + \kappa_4 \nabla \theta + \kappa_7 \mathcal{E} \times \mathbf{B} + \kappa_9 \nabla \theta \times \mathbf{B} + \kappa_{11} (\mathbf{B} \cdot \nabla \theta) \mathbf{B} + \kappa_{12} (\mathbf{B} \cdot \mathcal{E}) \mathbf{B} \quad (58)$$

As can be seen, the classical MHD modeling neglects many effects. Improvements could be made by including the -strictive effects of equation (54) as well as cross-effects between electric current and heat transfer.

Whereas the classical EHD model includes many of the important effects and matches the unified EMFD theory very well, classical MHD formulations could be improved. Depending on the problem being studied, improvements in the force, current and heat transfer

terms could be made. As in classical EHD modeling, it is important to be aware of the fact that many force, current and heat transfer terms can be written in several different formats, each of which is equivalent. It is therefore important to recognize the potential danger in simply adding terms from different MHD models.

5. CONCLUSION

The objective of this paper was to survey sufficient background material to allow initial implementation of a unified EMFD theory presented by Eringen and Maugin. To accomplish this the basics of electro-magnetic field theory was presented in Part 1 [1], with emphasis placed on describing the causes and effects of material polarization and magnetization. This paper presented the equations governing unified EMFD flows. The equations governing flow characteristics are contained in Maxwell's equations of electro-magnetic fields, the Navier–Stokes flow-field equations and the constitutive relations for current, heat transfer and material equation of state.

The essence of the paper is a presentation of classical models for EHD and MHD flows compared with the unified EMFD theory. Both classical models assumed material isotropy and linear constitutive theory which was shown often to be a valid assumption [6, 7]. It was shown that classical EHD models matched nearly identically with the EMFD theory. The classical equations for heat transfer were the only place where EHD and EMFD models did not match. Finally, it was shown that even though the EHD and EMFD models matched, they were often written in different forms, making them seem incompatible at first glance.

A similar comparison between the classical MHD model and the unified EMFD model was performed. Unlike the comparison with EHD, the unified model did not compare well with the classical model. Diamagnetophoretic and magnetostrictive terms were not included in classical MHD modeling of force. Being defined as first order effects [6, 7] these terms should be considered depending on the characteristics of the particular flow being analysed. Further, classical MHD theory was shown to neglect several cross-effect terms in the formulation of the electric current. Classical MHD theory did, however, include all first order effects in the electric current formulation.

Finally, because many of the terms in electro-magnetic force, electric current and heat transfer may be written in different forms, it is important to recognize the danger of simply adding of terms from classical models without fully understanding their meanings.

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