



A FAST ITERATIVE ALGORITHM FOR
 INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

SEUNGSOO LEE GEORGE S. DULIKRAVICH
 GRADUATE ASSISTANT ASSOCIATE PROFESSOR
 DEPARTMENT OF AEROSPACE ENGINEERING
 THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802



PEM
 PROGRAMA DE
 ENGENHARIA
 MECÂNICA
 COPPE/UFRJ

ABSTRACT

Based on an artificial compressibility method, new explicit Runge-Kutta time stepping algorithms were developed for steady, incompressible, Navier-Stokes equations for laminar and turbulent flows. Their accuracy has been tested against known analytic solutions. The codes are capable of computing two-dimensional and three-dimensional steady flows with high accuracy since they use characteristic boundary conditions. The codes have been successfully accelerated using our Distributed Minimal Residual (DMR) method. The accelerated algorithm offers substantial savings in the CPU time, especially in laminar problems when the grid is not extremely clustered.

INTRODUCTION

Unlike the algorithms for computations of compressible viscous flows, computational methods for incompressible viscous flows have been developed relatively slowly. The main difficulty which arises during the incompressible computation is the lack of a time derivative term in the continuity equation. One can set the free stream Mach number to extremely low value in existing compressible viscous flow codes in order to simulate the incompressible flow. At very low Mach number, however, the system of the equations becomes exceedingly stiff. Therefore, we are forced to use a different method to solve the incompressible Navier-Stokes equations.

One of the methods devised to circumvent this difficulty is the artificial compressibility concept originated by Chorin [1]. In this concept, an artificially time dependent derivative term $\frac{\partial(p/\beta)}{\partial t}$ is added to the continuity equation with a user specified control parameter β . The added term allows the use of the existing methods for compressible flow calculations. Since the artificial time derivative added to the continuity equation diminishes as the solution converges, the added term can be chosen so as to achieve the maximum convergence rate. Because of the added artificially time dependent term, it is difficult to apply this method to time accurate flow calculations. Either iteration on pressure (after solving momentum equation) or a larger value of β in modified solution vector can be used. As β becomes larger, the stiffness of the system of equations increases consequently, the first method is often used for the time accurate computations[2,3].

In this paper, our DMR(Distributed Minimal Residual) method[4,5,6] is incorporated in the artificial compressibility code in order to maximize the convergence rate. The DMR belongs to a general class of the extrapolation techniques in which the solution is updated using information from a number of consecutive time steps in such a way that the future residual is minimized. Unlike other similar methods, each component of the solution vector is updated using a

separate sequence of acceleration factors, which is equivalent to letting the each equation evolve at its own optimal convergence rate.

THE NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE FLOWS

After non-dimensionalization, the system of Navier-Stokes equations for the incompressible flow in conservative form can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = \frac{\partial}{\partial x} \left\{ S \frac{\partial Q}{\partial x} \right\} + \frac{\partial}{\partial y} \left\{ S \frac{\partial Q}{\partial y} \right\} + \frac{\partial}{\partial z} \left\{ S \frac{\partial Q}{\partial z} \right\} \quad (1)$$

where the generalized vectors for non-dimensional variables are

$$Q = \begin{bmatrix} 0 \\ u \\ v \\ w \end{bmatrix} \quad E = \begin{bmatrix} u \\ u^2 + p \\ uv \\ uw \end{bmatrix} \quad F = \begin{bmatrix} v \\ vu \\ v^2 + p \\ vw \end{bmatrix} \quad G = \begin{bmatrix} w \\ wu \\ wv \\ w^2 + p \end{bmatrix}$$

$$S = \frac{1}{Re} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (2)$$

Here, the velocity vector components are u,v and w and they act in the x,y and z direction, respectively. Notice that in the above equations, the density, ρ , is absorbed by the pressure, p, and the Reynolds number is denoted by Re.

Since the first component of the vector Q is zero, it is not possible to solve the system of equations simultaneously. If the vector Q is modified as shown below

$$Q = \begin{bmatrix} p \\ \beta \\ u \\ v \\ w \end{bmatrix} \quad (3)$$

then the system of equations can be solved simultaneously. The system is of mixed parabolic-hyperbolic type in time depending on the magnitude of the viscous term. Here, β is the artificial compressibility factor[1,2,3,6,7].

Coordinate Transformation. Consider the transformation from x, y, z Cartesian coordinate system to a general curvilinear ξ, η, ζ nonorthogonal coordinate system.

$$(x, y, z) \rightarrow (\xi, \eta, \zeta)$$

The equation (1) is transformed to

$$\frac{\partial Q'}{\partial t} + \frac{\partial E'}{\partial \xi} + \frac{\partial F'}{\partial \eta} + \frac{\partial G'}{\partial \zeta} = D^2(JQ') \quad (4)$$

where

$$Q' = \frac{1}{J}Q = \frac{1}{J} \begin{bmatrix} p \\ \beta \\ u \\ v \\ w \end{bmatrix}$$

$$E' = \frac{1}{J}(\xi_x E + \xi_y F + \xi_z G) = \frac{1}{J} \begin{bmatrix} U \\ Uu + \xi_{xp} \\ Uv + \xi_{yp} \\ Uw + \xi_{zp} \end{bmatrix}$$

$$F' = \frac{1}{J}(\eta_x E + \eta_y F + \eta_z G) = \frac{1}{J} \begin{bmatrix} V \\ Vu + \eta_{xp} \\ Vv + \eta_{yp} \\ Vw + \eta_{zp} \end{bmatrix}$$

$$G' = \frac{1}{J}(\zeta_x E + \zeta_y F + \zeta_z G) = \frac{1}{J} \begin{bmatrix} W \\ Wu + \zeta_{xp} \\ Wv + \zeta_{yp} \\ Ww + \zeta_{zp} \end{bmatrix} \quad (5)$$

and the viscous terms are contained in

$$D^2(JQ') = \left[\sum_{ij} g_{ij} (JQ')_{,i} \right]_{,j} \quad (6)$$

where g_{ij} is the contravariant metric tensor. Einstein summation convention is used for this expression.

$$g_{ij} = \begin{bmatrix} \nabla \xi \nabla \xi & \nabla \xi \nabla \eta & \nabla \xi \nabla \zeta \\ \nabla \eta \nabla \xi & \nabla \eta \nabla \eta & \nabla \eta \nabla \zeta \\ \nabla \zeta \nabla \xi & \nabla \zeta \nabla \eta & \nabla \zeta \nabla \zeta \end{bmatrix} \quad (7)$$

$U, V,$ and W are the contravariant velocity vector components acting in $\xi, \eta,$ and ζ direction respectively. They are defined as

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (8)$$

and J is the determinant of Jacobian matrix (J).

$$J = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix}^{-1} \quad (9)$$

NUMERICAL ALGORITHM

Runge-Kutta Time Stepping Algorithm. Runge-Kutta time stepping algorithm and the spatial discretization is performed by defining the residual vector R as

$$R = \frac{\partial E'}{\partial \xi} + \frac{\partial F'}{\partial \eta} + \frac{\partial G'}{\partial \zeta} - D^2(JQ') \quad (10)$$

The Runge-Kutta algorithm can be written as

$$Q^0 = Q^i$$

$$\Delta Q^k = -\alpha_k \Delta t R^{k-1} \quad k = 1, 2, \dots, K \quad (11)$$

$$Q^{i+1} = Q^i + \Delta Q^K$$

where α_k are the coefficients for each of the K stage Runge-Kutta scheme required to advance the solution from the time level t to the time level $t+1$ [8,9]. For example, $\alpha_k = 1/4, 1/3, 1/2$ and 1.0 for the four stage Runge-Kutta scheme.

To reduce the computational effort, we calculate the viscous part of the residual once every time level. This does not interfere with the stability of the time stepping algorithm.

Spatial Discretization. The residual of the Navier-Stokes equations (10) is discretized by the second order central difference approximation. For example, the first derivative is approximated as

$$\frac{\partial E'}{\partial \xi} = \frac{E'_{i+1/2k} - E'_{i-1/2k}}{2\Delta \xi} \quad (12)$$

The second derivative is discretized as

$$\frac{\partial}{\partial \xi} \left[\frac{\partial (JQ')}{\partial \xi} \right] = \frac{1}{\Delta \xi^2} \left[\sum_{j=1/2}^S \right]_{i+1/2jk} \{ (JQ')_{i+1jk} - (JQ')_{ijk} \} - \frac{1}{\Delta \xi^2} \left[\sum_{j=1/2}^S \right]_{i-1/2jk} \{ (JQ')_{ijk} - (JQ')_{i-1jk} \} \quad (13)$$

The mixed derivative is discretized as

$$\frac{\partial}{\partial \xi} \left[\frac{\partial (JQ')}{\partial \eta} \right] = \frac{1}{4\Delta \xi \Delta \eta} \left[\sum_{j=1/2}^S \right]_{i+1/2jk} \times \{ (JQ')_{i+1j+1k} - (JQ')_{i+1j-1k} + (JQ')_{ij+1k} - (JQ')_{ij-1k} \} - \frac{1}{4\Delta \xi \Delta \eta} \left[\sum_{j=1/2}^S \right]_{i-1/2jk} \times \{ (JQ')_{ij+1k} - (JQ')_{ij-1k} + (JQ')_{i-1j+1k} - (JQ')_{i-1j-1k} \} \quad (14)$$

Artificial Dissipation. It is well known that the computational results obtained with central difference schemes suffer from even-odd decoupling resulting in severe numerical oscillations. These oscillations can be eliminated either by using the staggered grid (Patankar[9]) or by adding small amount of artificial dissipation to the discretized equation. The artificial dissipation used in this paper was originally suggested by Steger[11] and given by

$$D(JQ') = \frac{\epsilon}{8J\Delta t} \nabla^4 [JQ'] \quad (15)$$

where ∇^4 denotes the fourth order elliptic operator and ϵ is the user specified control parameter.

The residual defined in equation (10) is modified with the artificial dissipation. The amount of artificial dissipation is controlled by the stability of the system. With the linear stability analysis, one can determine the maximum allowable value of ϵ . Like the viscous terms, the artificial dissipation terms are recomputed once every time level and kept constant during every intermediate stage of the Runge-Kutta scheme.

EIGENVALUE ANALYSIS

The Jacobians of the inviscid part of the Navier-Stokes equations in Cartesian coordinates are given as

$$A = \frac{\partial E}{\partial Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta & 2u & 0 & 0 \\ 0 & v & u & 0 \\ 0 & w & 0 & u \end{bmatrix} \quad B = \frac{\partial F}{\partial Q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & v & u & 0 \\ \beta & 0 & 2v & 0 \\ 0 & 0 & w & v \end{bmatrix}$$

$$C = \frac{\partial G}{\partial Q} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & w & 0 & u \\ 0 & 0 & w & v \\ \beta & 0 & 0 & 2w \end{bmatrix} \quad (16)$$

The Jacobians in the generalized curvilinear coordinates are easily obtained from the defined in Cartesian coordinates. Thus,

$$A' = \frac{\partial E'}{\partial Q'} = K(U, \xi_x, \xi_y, \xi_z)$$

$$B' = \frac{\partial F'}{\partial Q'} = K(V, \eta_x, \eta_y, \eta_z)$$

$$C' = \frac{\partial G'}{\partial Q'} = K(W, \zeta_x, \zeta_y, \zeta_z) \quad (17)$$

where

$$K(k_1, k_2, k_3) = k_1 \frac{\partial E}{\partial Q} + k_2 \frac{\partial F}{\partial Q} + k_3 \frac{\partial G}{\partial Q}$$

$$= \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ \beta k_1 & k+k_1 u & k_2 u & k_3 u \\ \beta k_2 & k_1 v & k+k_2 v & k_3 v \\ \beta k_3 & k_1 w & k_2 w & k+k_3 w \end{bmatrix} \quad (18)$$

and

$$k = k_1 u + k_2 v + k_3 w \quad (19)$$

The eigenvalues of the matrix K are given by

$$\Lambda = \text{diag}(k - c, k + c, k, k) \quad (20)$$

where the equivalent speed of sound, c, is given as

$$c = \sqrt{k^2 + \beta(k_1^2 + k_2^2 + k_3^2)} \quad (21)$$

It is noticeable that one of the eigenvalues is negative; thus, the incompressible flow is "subsonic", and care should be exercised in applying the boundary conditions.

STABILITY ANALYSIS

The time increment of the explicit scheme is bounded by the stability conditions. We will define the time step by considering hyperbolic part of equation and parabolic part of equation separately and by combining these time steps [12]. The system becomes of hyperbolic type when viscosity is neglected. Then, the stability bound of the resulting system is determined by the CFL (Courant-Friedrichs-Lewy) number. The maximum time steps for each of the coordinate directions are defined by

$$\Delta t_{h\xi} = \frac{CFL}{U + c_\xi} \quad \Delta t_{h\eta} = \frac{CFL}{V + c_\eta} \quad \Delta t_{h\zeta} = \frac{CFL}{W + c_\zeta}$$

and the combined maximum time step for the hyperbolic part of the system is defined by

$$\Delta t_h = \frac{\Delta t_{h\xi} \Delta t_{h\eta} \Delta t_{h\zeta}}{\Delta t_{h\xi} \Delta t_{h\eta} + \Delta t_{h\eta} \Delta t_{h\zeta} + \Delta t_{h\zeta} \Delta t_{h\xi}} \quad (22)$$

When the convective part of the acceleration is neglected, the system becomes of parabolic type. The stability of the parabolic type system is dictated by the non-dimensional number σ (von Neumann number).

For each direction, the maximum time steps are defined by

$$\Delta t_{p\xi} = \frac{\sigma}{g_{11} Re} \quad \Delta t_{p\eta} = \frac{\sigma}{g_{22} Re} \quad \Delta t_{p\zeta} = \frac{\sigma}{g_{33} Re}$$

and the combined maximum time step for the parabolic part is given by

$$\Delta t_p = \frac{\Delta t_{p\xi} \Delta t_{p\eta} \Delta t_{p\zeta}}{\Delta t_{p\xi} \Delta t_{p\eta} + \Delta t_{p\eta} \Delta t_{p\zeta} + \Delta t_{p\zeta} \Delta t_{p\xi}} \quad (23)$$

The total maximum time step is estimated conservatively [12] as

$$\Delta t = \frac{\Delta t_h \Delta t_p}{\Delta t_h + \Delta t_p} \quad (24)$$

The artificial compressibility method with Runge-Kutta time stepping is stable with CFL number less than approximately 2.8.

BOUNDARY CONDITIONS

Though we solve the viscous flow, the viscous effects are dominant only in the vicinity of the wall. Therefore, for the purpose of enforcing boundary conditions it is natural to assume that the flow is inviscid at the inlet and the exit plane. After neglecting the viscous terms, the system of equations is hyperbolic in time at the inlet and exit. For the compatibility condition, therefore, we have to consider the direction of the characteristics. As stated earlier, the incompressible flow is "subsonic". Thus one equation of motion should be considered with three boundary conditions at the inlet. At the exit, three equations with one boundary condition need to be applied. At the inlet we specify U , $\frac{V}{U}$ and $\frac{W}{U}$, while the back pressure p is specified at the exit. Therefore, we can easily specify the swirl as the inlet boundary condition. Also, the flow is assumed to be locally one-dimensional at the inlet and exit boundary in order to transform the equation into the characteristic form.

The similarity transform matrix for three dimensional case can be derived from the Jacobian matrix K in the general coordinates, and given as

$$M^{-1} = \begin{bmatrix} \frac{k+c}{2c^2} & \frac{k_1}{2c^2} & \frac{k_2}{2c^2} & \frac{k_3}{2c^2} \\ -\frac{k-c}{2c^2} & \frac{k_1}{2c^2} & \frac{k_2}{2c^2} & \frac{k_3}{2c^2} \\ \frac{(k^2-c^2)v+\beta k k_2}{c^2} & \frac{k_1(kv+\beta k_2)}{c^2} & \frac{k_2(kv+\beta k_2)}{c^2} & \frac{k_3(kv+\beta k_2)}{c^2} \\ \frac{(k^2-c^2)w+\beta k k_3}{c^2} & \frac{k_1(kw+\beta k_3)}{c^2} & \frac{k_2(kw+\beta k_3)}{c^2} & \frac{k_3(kw+\beta k_3)}{c^2} \end{bmatrix} \quad (25)$$

Premultiplying the equations by M_ξ^{-1} (in ξ direction) results in the characteristic form of equations. The equation corresponding to the negative eigenvalue at the inlet is to be selected, while at the exit the equations corresponding to positive eigenvalues are chosen. This selection procedure can be thought of as a matrix operation, and we designate the operator as L. If the boundary condition is given by Ω , then

$$\Omega^{i+1} = \Omega^i + \frac{\partial \Omega}{\partial Q} \Delta Q^i$$

or

$$\frac{\partial \Omega}{\partial Q} \Delta Q^i = 0 \quad (26)$$

and the equation (24) is added to the transformed-selected equations, that is,

$$\left[LM_{\xi}^{-1} + \frac{\partial \Omega}{\partial Q} \right] \Delta Q^i = -\alpha_k M_{\xi}^{-1} R^{k-1} \quad (27)$$

This boundary condition treatment is applied at every stage of each Runge-Kutta step.

At the solid wall the boundary condition is fairly simple. The contravariant velocities U, V and W are set to zero. The pressure is linearly extrapolated from the nearest grid point, which is acceptable if the grid is nearly normal to the wall.

THE DISTRIBUTED MINIMAL RESIDUAL (DMR) METHOD

The DMR formulation for acceleration of iterative solutions of a system of partial differential equations takes account of inviscid terms, physically viscous terms and artificially dissipative terms.

The residual at time level t is given by

$$R^t = \frac{\partial E^t}{\partial \xi} + \frac{\partial F^t}{\partial \eta} + \frac{\partial G^t}{\partial \zeta} - D^2(JQ^t) + D(JQ^t) \quad (28)$$

where E', F', G' and Q' are evaluated at the time level t. Similarly, the residual at time level t+1 is

$$R^{t+1} = \frac{\partial E^t}{\partial \xi} + \frac{\partial F^t}{\partial \eta} + \frac{\partial G^t}{\partial \zeta} - D^2(JQ^t) + D(JQ^t) \quad (29)$$

where E', F', G' and Q' are evaluated at the time level t+1.

Assume that the solution at time level t+1 is extrapolated from the previous M time levels. Then, we [4,5,6] can say that

$$Q^{t+1} = Q^t + \sum_{m=1}^M \Theta^m \quad (30)$$

where

$$\Theta^m = \begin{bmatrix} \omega_1^m \Delta_1^m \\ \omega_2^m \Delta_2^m \\ \vdots \\ \omega_L^m \Delta_L^m \end{bmatrix} \quad (31)$$

Here, ω 's are the acceleration factors to be calculated, Δ 's are the corrections computed with the original scheme, the M denotes the number of consecutive time steps combined, and L denotes the number of equations in the system. For example, L=4 for the incompressible three-dimensional flow problem accounting for one mass conservation equation and three momentum equations.

Using Taylor series expansion and truncating the terms which are higher than second order in Δt , the equation (29) becomes approximately

$$R^{t+1} = R^t + \sum_{m=1}^M \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \Theta^m \quad (32)$$

The global residual is defined as

$$R^i = \sum_D R^{i*} R^i \quad (33)$$

where \sum_D denotes summation over the computational domain, and

(.)^{*} means transpose of a vector. Then the residual at time level t+1 will be

$$R^{t+1} = \sum_D \left\{ R^t + \sum_{n=1}^M \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \Theta^n \right\}^* \left\{ R^t + \sum_{m=1}^M \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \Theta^m \right\} \quad (34)$$

In order to minimize the R^{t+1} , we can choose the ω 's from the following equations

$$\frac{\partial R^{t+1}}{\partial \omega_r^m} = 0 \quad (35)$$

that is,

$$-\sum_D R^{t*} \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \frac{\partial \Theta^m}{\partial \omega_r^m}$$

$$= \sum_D \sum_{n=1}^M \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \Theta^n$$

$$\cdot \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \frac{\partial \Theta^m}{\partial \omega_r^m} \quad (36)$$

where

$$\frac{\partial \Theta^m}{\partial \omega_r^m} = \Delta_r^m \delta_r^m \quad (37)$$

Here δ_r^m is defined by

$$\delta_r^m = \text{col}[\delta_{qr}] \quad (38)$$

and δ_{qr} is the Kronecker delta. From the equation (31), we know that

$$\Theta^n = \sum_{q=1}^L \omega_q^n \frac{\partial \Theta^n}{\partial \omega_q^n} \quad (39)$$

Notice that $\frac{\partial \Theta^n}{\partial \omega_q^n}$ is not a function of ω 's.

$$\text{Let } a_q^n = \left[\frac{\partial}{\partial \xi} A^t + \frac{\partial}{\partial \eta} B^t + \frac{\partial}{\partial \zeta} C^t - D^2 J + DJ \right] \frac{\partial \Theta^n}{\partial \omega_q^n} \quad (40)$$

Then the equation (36) becomes

$$-\sum_D R^{t*} a_r^m = \sum_{n=1}^M \sum_{q=1}^L \sum_D \omega_q^n a_q^n a_r^m \quad (41)$$

For the simplicity,

$$\text{let } c_{qr}^{nm} = \sum_D a_q^n a_r^m \quad (42)$$

$$b_r^m = -\sum_D R^{t*} a_r^m \quad (43)$$

Then equation (41) can be written as

$$\sum_{n=1}^M \sum_{q=1}^L \omega_q^n c_{qr}^{nm} = b_r^m \quad (44)$$

representing the system of $M \times L$ linear algebraic equations for the optimum acceleration factors ω . For example, if we are to combine M = 2 consecutive global Runge-Kutta time steps to extrapolate the solution and to solve the three-dimensional incompressible Navier-

Stokes equations ($L = 4$), we need to solve 8 equations for 8 ω 's every 30 or so time steps. Using a different sequence of acceleration factors for each component of the solution vector is equivalent to using a different time step for each equation, which allows acceleration of the scheme.

RESULTS

Based on the theory outlined in this paper, explicit analysis codes for two-dimensional and three-dimensional flows were developed. We used $CFL=2.8$, $\sigma = 0.1$ and $\epsilon = 0.05$ in all test cases.

In the case of a U-shaped duct we used 171×31 grid, $Re=400$ and a uniform inlet velocity profile. The computational results indicate large separated flow region in the outlet (Fig.1) and strong pressure gradients on the inner wall in the bend region (Fig.2). Fig.3 and Fig.4 demonstrate that the DMR algorithm effectively accelerates this type of iteration and saves approximately 40% of CPU time.

The same code was tested for a case of a two-dimensional non-staggered cascade of NACA 0012 airfoils (Fig.5) at zero free stream angle. Uniform inlet velocity field changes (Fig.5) to a thick boundary layer type flow typical of the low Reynolds numbers ($Re=500$). The DMR offers very high savings (75% reduction in the number of iterations) in this test case (Fig.6) which used clustered 20×40 grid.

The three-dimensional code was first tested against an analytic result (Fig.7) for a fully developed flow in a straight duct having square cross section. Starting with the uniform inlet velocity profile, the computed axial components of the velocity vector obtained after 5000 iterations on a $30 \times 21 \times 21$ grid are shown in Fig.8 indicating high accuracy of the code.

In the case of an L-shaped duct [14] having square cross sections (Fig.9) and a fully developed inlet velocity profile, we used $Re=790$ and $51 \times 31 \times 31$ mildly clustered grid. The computed flow in the cross section planes at $\theta = 30^\circ$ (Fig.10), $\theta = 60^\circ$ (Fig.11) and $\theta = 90^\circ$ (Fig.12) of the duct compare very well with other published results [14].

SUMMARY

The new Distributed Minimal Residual (DMR) algorithm has been applied to laminar, two-dimensional, incompressible, steady flows with success. In the case of high Reynolds number flows (we have duplicated the results [13] for the U-shaped channel with $Re=10$ million), the flow is turbulent and very strong grid clustering at the solid boundaries is required. Because of the excessively high aspect ratios of the grid cells in turbulent flow computations, DMR does not offer noticeable computing time savings.

ACKNOWLEDGEMENTS

This work was supported by a research grant from the AFOSR/NM. Computations were performed on NAS facility at NASA Ames Research Center.

REFERENCES

- [1] Chorin, A. J., "A Numerical Method for Solving Incompressible Viscous Flow Problems," *Journal of Computational Physics*, Vol. 2, 1967, pp. 12-26.
- [2] Kwak, D., Chang, J. L. C., Shanks, S. P. and Chakravarthy, S. R., "An Incompressible Navier-Stokes Flow Solver in Three Dimensional Curvilinear Coordinate System Using Primitive Variables," *AIAA Journal*, Vol. 24, No. 3, pp. 390-396.
- [3] Athavale, M. and Merkle, C. L., "An Upwind Differencing Scheme for Time-Accurate Solutions of Unsteady Incompressible Flow," *AIAA paper 88-3650-CP*, Cincinnati, OH, July 1988.
- [4] Dulikravich, G. S., Dorney, D. J. and Lee, S., "Iterative Acceleration and Physically Based Dissipation for Euler Equations of Gasdynamics," *Symposium on Advances and Applications in Computational Fluid Dynamics*, ASME Winter Annual Meeting, Chicago, IL, November 28-December 2, 1988.
- [5] Lee, S. and Dulikravich, G. S., "Acceleration of Rational Runge-Kutta Algorithms for Euler Equations of Gasdynamics," *AIAA paper 89-0097* presented at the AIAA Aerospace Science Meeting, Reno, NV, January 1989.
- [6] Lee, S. and Dulikravich, G. S., "Accelerated Computation of Viscous, Steady, Incompressible Flows," *ASME paper 89-GT-45*, presented at the International Gas Turbine Conference, Toronto, Canada, June 5-8, 1989.
- [7] Choi, D. and Merkle, C. L., "Application of Time-Iterative Schemes to Incompressible Flow," *AIAA Journal*, Vol. 23, No.10, October 1985, pp. 1518-1524.
- [8] Jameson, A., "Numerical Solutions of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time-Stepping Scheme," *AIAA paper 81-1259*, *AIAA 14th Fluid and Plasma Dynamics Conference*, June 23-25, 1981, Palo Alto, CA.
- [9] Merkle, C. L. and Tsai, Y.-L., "Application of Runge-Kutta Schemes to Incompressible Flows," *AIAA paper 86-0553*, Reno, NV, January 1986.
- [10] Patankar, S. Y., *Numerical Heat Transfer and Fluid Flow*, Hemisphere Publishing Co., New York, 1980.
- [11] Steger, J. L., "Implicit Finite-Difference Simulation of Flow About Arbitrary Two-Dimensional Geometries," *AIAA Journal*, Vol. 16, No. 7, July 1987, pp. 679-686.
- [12] MacCormack, R. W. and Baldwin, B. S., "A Numerical Method for Solving the Navier-Stokes Equations With Application to Shock-Boundary Layer Interaction," *AIAA paper 75-1*, January 1975.
- [13] Monson, D. J., Seegmiller, H. L. and McConnaughey, P. K., "Comparison of LDV Measurements and Navier-Stokes Solutions in a Two-Dimensional 180-Degree Turn-Around Duct," *AIAA paper 89-0275*, Reno, Nevada, 1989.
- [14] Rogers, S. E., Kwak, D. and Kiris, C., "Numerical Solution of the Incompressible Navier-Stokes Equations for Steady-State and Time-Dependent Problems," *AIAA paper 89-0463*, Reno, Nevada, 1989.

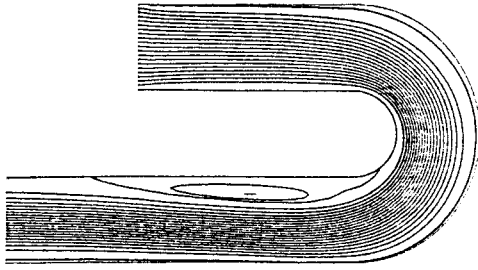


Fig. 1 Streamlines for a separated flow in a U-shaped channel
Re=400.

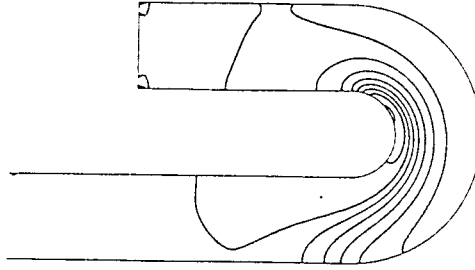


Fig. 2 Isobars for a separated flow in the U-shaped channel
Re=400.

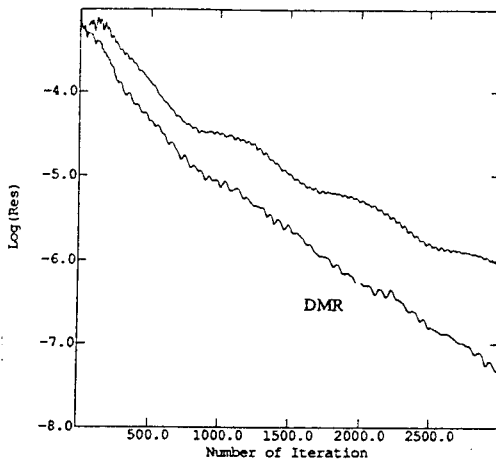


Fig. 3 Convergence history for the U-shaped channel residual vs. no. of iterations

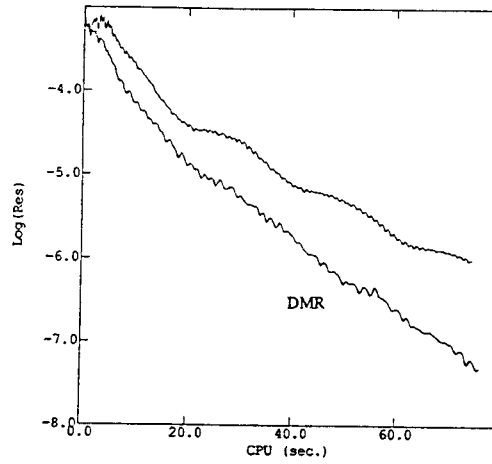


Fig. 4 Convergence history for the U-shaped channel residual vs. CPU time

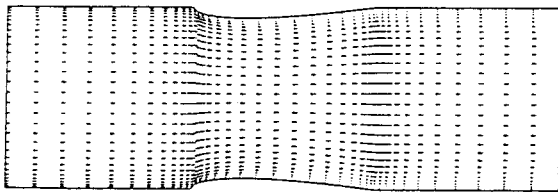


Fig. 5 Velocity vector field for a nozzle flow ; Re=500.

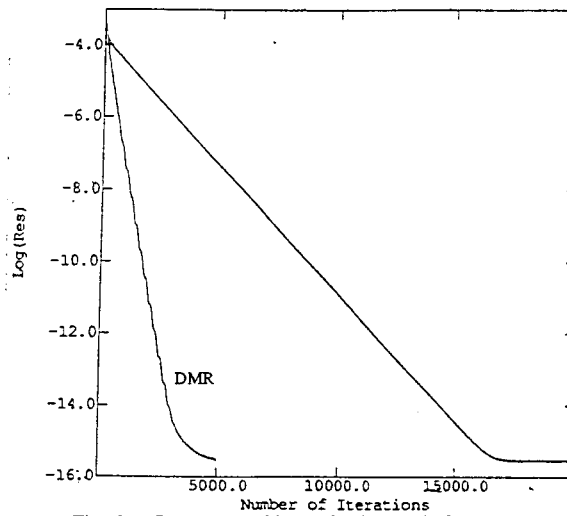


Fig. 6 Convergence history for the nozzle flow residual vs. no. of iterations

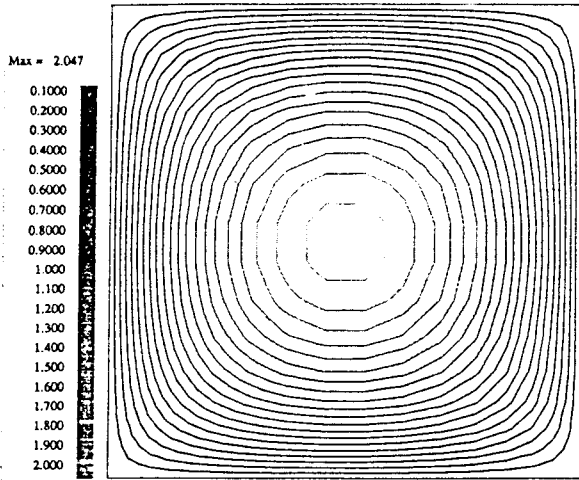


Fig. 7 Analytic values of const. u-velocity in the square cross section straight duct

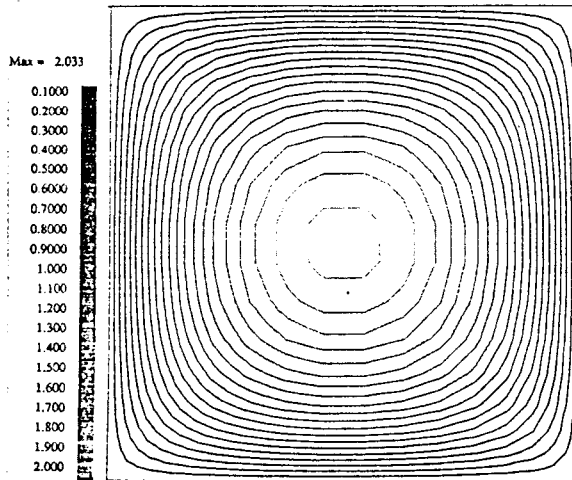


Fig. 8 Computed values of const. u-velocity in the square cross section straight duct

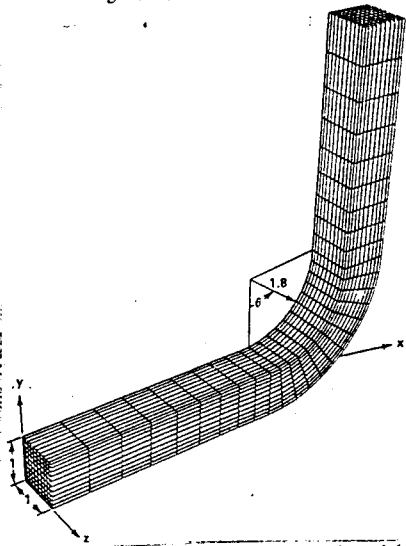


Fig. 9 Computational grid for an L-shaped duct

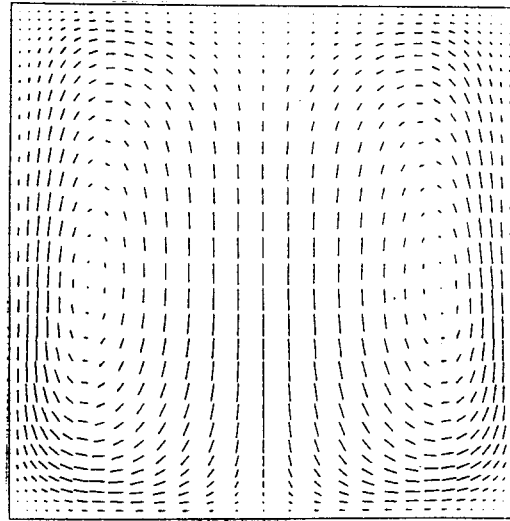


Fig.10 Cross flow at 30° in the L-shaped duct

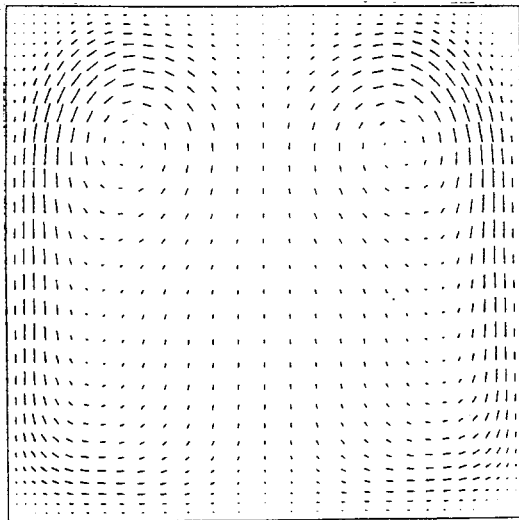


Fig.11 Cross flow at 60° in the L-shaped duct

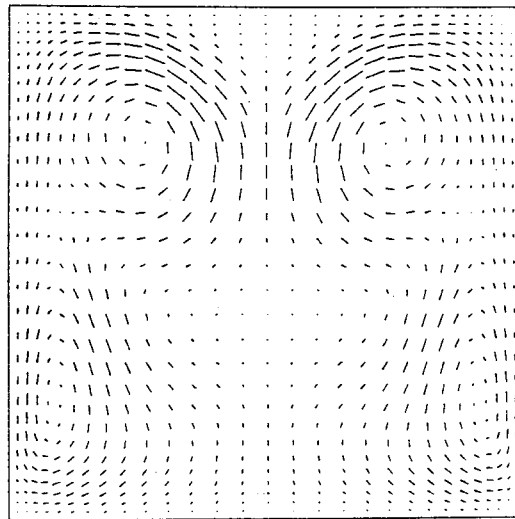


Fig.12 Cross flow at 90° in the L-shaped duct