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PHYSICALLY BASED ARTIFICIAL DISSIPATION CONCEPTS IN COMPUTATIONAL FLUID DYNAMICS

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ABSTRACT

A summary of recent work on analysis of conventional forms of artificial dissipation in computational fluid dynamics is presented. It is shown that they introduce extraneous higher order non-physical terms that are often of vastly different magnitude and opposite sign than the higher order terms that exist in more complete physical models. Several new artificial dissipation models are suggested that are consistent with the physically dissipative analytic models for the flow field. The new models offer viable alternatives to the existing artificial dissipation models since they implicitly satisfy the second law of thermodynamics and produce results that are comparable to the existing models.

1. INTRODUCTION

When analytic models of fluid dynamics are integrated numerically, the oscillations can be damped by either explicitly or implicitly adding a certain amount of artificial dissipation. Most of the existing artificial dissipation formulations are intuitive in the sense that a proper combination of artificial dissipation terms can lead to fast convergence and seemingly accurate results, while an improper combination of artificial dissipation terms can generate incorrect results and reduced convergence rates. The intuitive formulations often locally violate the second law of thermodynamics [1,2] so that, for example, separated flow regions produced in transonic shocked flow definitely (Fig. 1) depend on the magnitude and the form of artificial dissipation [3,4,5]. It can be concluded that the intuitive formulations for artificial dissipation have not been fully analyzed and that their accuracy and control are still open questions [6]. Only recently have attempts been made to create new formulations for the artificial dissipation that are consistent with the physics of the more complete analytical models of the flow field.

2. THE FULL POTENTIAL EQUATION (FPE)

An exact, nondissipative analytical model for irrotational, inviscid, homentropic flow of a calorically perfect gas is the Full Potential Equation (FPE) which can be expreesed in terms of an (s,n,m) locally streamline aligned coordinate system [7] as

$$\rho \{[(1-M^2)\phi_{ss} + \phi_{mm} + \phi_{nn}] - \frac{1}{a^2} (\phi_{tt} + 2\phi_t \phi_{st})\} = 0 \qquad (1)$$

where t, ρ, M, a, and ϕ are time and local values of density, Mach number, speed of sound, and potential function, respectively. The most popular numerical schemes for integration of this equation are based on Artificial Density [8] or Artificial Viscosity [7] concept (ADV) of generating the artificial dissipation in a locally supersonic region by altering the density according to

$$\tilde{\rho} = \rho - C\tilde{\mu}\, \rho_s \qquad (2)$$

Here, C is a user specified constant and $\tilde{\mu}$ is a switching function [7] most commonly defined as

$$\tilde{\mu} = \max \{0; (1 - \frac{M_c^2}{M^2})\} \qquad (3)$$

where M_c = const is a user specified cutoff Mach number. The ADV formulations effectively [9] result in a modified steady state FPE (in two-dimensions) that is of the form:

$$\nabla \cdot (\tilde{\rho}\nabla\phi) = \rho \{[(1-M^2)\, \phi_{ss} + \phi_{nn}] + C(M^2-M_c^2)\phi_{sss} + C(M^2-M_c^2) \frac{\phi_{ss}\phi_{nn}}{\phi_s}$$

$$+ C\, [(M^2-M_c^2)(2-(2-\gamma)M^2) + \frac{\gamma-1}{a^2} M_c^2] \frac{\phi_{ss}^2}{\phi_s}\} \qquad (4)$$

This formulation was shown [9,10] to generate non-symmetric shock profiles excessive artificial entropy (Fig. 2) "checker-board" shock pattern, and near-shock oscillations.

A more complete analytical model for irrotational, non-isentropic, physically dissipative flows is known as the Physically Dissipative Potential (PDP) equation [11].

$$\rho\,\{[(1-M^2)\,\phi_{ss} + \phi_{mm} + \phi_{nn}] - \frac{1}{a^2}(\phi_{tt} + 2\phi_s\phi_{st})\} = \frac{\mu''}{Re}\{\frac{\phi_s}{a^2}(1 + \frac{\gamma-1}{P_r''})(\phi_{sss} + \phi_{smm} + \phi_{snn})$$

$$+ \frac{\gamma-1}{a^2}(1 - \frac{1}{P_r''})(\phi^2_{ss} + \phi^2_{mm} + \phi^2_{nn}) + 2\frac{\gamma-1}{a^2}(1 - \frac{2}{\mu''})(\phi_{ss}\phi_{mm} + \phi_{ss}\phi_{nn} + \phi_{mm}\phi_{nn})$$

$$- 2\frac{\gamma-1}{a^2}(\frac{1}{P_r''} - \frac{2}{\mu''})(\phi^2_{sm} + \phi^2_{sn} + \phi^2_{mn}) - \frac{1}{a^2}(\phi_{sst} + \phi_{mmt} + \phi_{nnt})\} \tag{5}$$

This is an exact formula for non-dimensional mass conservation (left-hand-side) that also satisfies the exact momentum and energy conservation for a calorically perfect gas with physical dissipation due to molecular viscosity and heat conductivity (right-hand-side). Here, Re, μ'', λ, Pr", γ are the Reynolds number, nondimensional longitudinal viscosity coefficient ($\mu'' = 2+\lambda/\mu$), secondary viscosity coefficient ($\lambda = -2/3$), longitudinal Prandtl number Pr" = $P_r\mu''$, and ratio of specific heats, respectively. When comparing the coefficients multiplying ϕ_{sss}, $\phi_{ss}\phi_{nn}$ and $\phi_{ss}^{\;2}$ terms [9] in the ADV and in the PDP formulation, it is obvious that they are not of the same magnitude and are often of the opposite sign. Notice that ADV generates an equivalent entropy that is significantly greater than the entropy generated by the PDP (Fig. 2).

Several alternative models for the artificial dissipation in the FPE have been derived and tested [9,10] that are based on the exact matching of certain coefficients in the PDP and in the generalized ADV. Specifically, Artificial Mass Flux (AMF) formulation [9] generates modified density of the form

$$\tilde{\rho} = \rho - \{\frac{1}{Re}\,\frac{[2+(\gamma-1)/P_r]2(\gamma-1)}{1+2(\gamma-1)}\,\frac{1}{\rho\,\phi_s}\}\rho_s \tag{6}$$

The other physically based artificial dissipation model is [10] the variable M_c and C formulation (MCC) which states that the constants M_c and C in Eq. 2 and 3 should be variable:

$$M_c^{\;2} = \frac{M^2\{(1 + \frac{\gamma-1}{P_r''})[2-(2-\gamma)M^2] \quad + (\gamma-1)(1 - \frac{1}{P_r''})\}}{(1 + \frac{\gamma-1}{P_r''})[2-(2-\gamma)M^2 - \frac{\gamma+1}{a^2}] \quad - (\gamma-1)(1 - \frac{1}{P_r''})}\;;\; C = \frac{(1 + \frac{\gamma-1}{P_r''})\,\frac{\phi_s}{a^2}\mu''}{\rho\,(M^2-M_c^{\;2})\;Re} \tag{7}$$

Both AMF and MCC physically based artificial dissipation formulations need only Reynolds number as the input parameter. Notice that the AMF formulation produces considerably lower values of equivalent entropy (Fig. 2) than the standard ADV formulation.

3. EULER EQUATIONS OF INVISCID GASDYNAMICS

The two-dimensional Euler equations in conservative form and non-orthogonal curvilinear boundary-conforming (ξ,η) coordinates can be expressed as

$$Q_t + E_\xi + F_\eta = 0 \tag{8}$$

where

$$Q = \frac{1}{J}\begin{bmatrix}\rho \\ \rho u \\ \rho v \\ \rho e_o\end{bmatrix} \qquad E = \frac{1}{J}\begin{bmatrix}\rho U \\ \rho u U + p\xi_x \\ \rho v U + p\xi_y \\ \rho h_o U\end{bmatrix} \qquad F = \frac{1}{J}\begin{bmatrix}\rho V \\ \rho u V + p\eta_x \\ \rho v V + p\eta_y \\ \rho h_o V\end{bmatrix} \tag{9}$$

where ρ,p,u,v,e_o and h_o are the non-dimensional values of local density, thermodynamic pressure, x-component of velocity, y-component of velocity, total mass-specific energy and total mass-specific enthalpy, respectively. For a calorically perfect gas

$$p = (\gamma-1)\ (\rho\, e_o - \frac{1}{2}\frac{(\rho\, u)^2}{\rho} + \frac{(\rho\, v)^2}{\rho})) \tag{10}$$

where γ represents the ratio of specific heats. The total mass-specific enthalphy, h_o, is $h_o = e_o + p/\rho$ and the determinant of the Jacobian is $J = \xi_x\eta_y - \eta_x\ \xi_y$.
The contravariant components U,V of the velocity vector are related to the Cartesian components u,v as follows

$$U = u\xi_x + v\xi_y = J(uy_\eta - vx_\eta); \qquad V = u\eta_x + v\eta_y = J(-uy_\xi + \eta x_\xi) \tag{11}$$

Numerical schemes presently used to integrate the Euler equations of inviscid gasdynamics rely heavily on the explicit addition of second order and fourth order artificial (non-physical) dissipation terms [12]. The second order terms are used to damp oscillations in shock regions, while the fourth order terms ensure monotonic convergence to steady state in smooth flow regions. A typical stage of the finite volume multistage Runge-Kutta time-stepping scheme for the Euler equations [12] is

$$Q^{(n)} = Q^{(o)} + \alpha_n\ \Delta t\ (E_\xi^{(n-1)} + F_\eta^{(n-1)} + D) \tag{12}$$

where D is the artificial dissipation [12] given as

$$D_{i,j} = (D^\xi_{i+1/2,j} - D^\xi_{i-1/2,j}) + (D^\eta_{i,j+1/2} - D^\eta_{i,j-1/2}) \tag{13}$$

Calculation of the artificially dissipative terms is done similarly [12] for all equations.

$$D^\xi_{i+m/2,j} = \epsilon^{(2)}_{i+m/2,j}\ Q^\xi_{i+m/2,j} - \epsilon^{(4)}_{i+m/2,j}\ (\theta_{i+m,j}\ Q^{\xi\xi}_{i+m,j} - \theta_{i,j}\ Q^{\xi\xi}_{i,j}) \tag{14}$$

$$D^\eta_{i,j+m/2} = \epsilon^{(2)}_{i,j+m/2}\ Q^\eta_{i,j+m/2} - \epsilon^{(4)}_{i,j+m/2}\ (\theta_{i,j+m}\ Q^{\eta\eta}_{i,j+m} - \theta_{i,j}\ Q^{\eta\eta}_{i,j}) \tag{15}$$

where $m = \pm 1$. The remaining terms are defined as follows:

$$\theta_{i,j} = \frac{1}{(J\Delta t)_{i,j}}\ ;\ Q^\xi_{i+m/2,j} = Q_{i+m,j} - Q_{i,j} \tag{16}$$

$$Q^{\xi\xi}_{i+m,j} = Q_{i+2m,j} - 2Q_{i+m,j} + Q_{i,j}\ ;\ Q^{\xi\xi}_{i,j} = Q_{i+1,j} - 2Q_{i,j} + Q_{i-1,j} \tag{17}$$

with similar expressions for the other terms of this type. The coefficients of second and fourth order artificial dissipation are defined [12], respectively, as

$$\epsilon^{(2)}_{i+m/2,j} = \theta_{i,j}\ \delta^{(2)}\ \max\ (p^{\xi\xi}_{i,j}\ ;\ p^{\xi\xi}_{i+m,j}) \tag{18}$$

$$\epsilon^{(4)}_{i+m/2,j} = \max\left(0;\ \delta^{(4)} - \delta^{(2)}\max\ (p^{\xi\xi}_{i,j}\ ;\ p^{\xi\xi}_{i+m,j})\right) \tag{19}$$

with similar expressions for the other terms of this type, where $\delta^{(2)} = 1/4$ to $1/2$ and

$\delta^{(4)} = 1/128$ to $1/64$ are typical constants [12]. Here, the local "directional dissipation sensors" are based on the second derivatives of pressure

$$p^{\xi\xi}_{ij} = \left|\frac{p_{i+1j} - 2p_{ij} + p_{i-1j}}{p_{i+1j} + 2p_{ij} + p_{i-1j}}\right| ; \qquad p^{\eta\eta}_{ij} = \left|\frac{p_{ij+1} - 2p_{ij} + p_{ij-1}}{p_{ij+1} + 2p_{ij} + p_{ij-1}}\right| \tag{20}$$

This type of artificial dissipation formulation effectively adds the following higher order non-physical terms to the Euler equations:

$$D \cong p_{\xi\xi}\,(Q_{\xi\xi} - Q_{\xi\xi\xi}) + p_{\xi\xi\xi}\,(Q_{\xi} - Q_{\xi\xi\xi}) + p_{\eta}\,(Q_{\eta\eta} - Q_{\eta\eta\eta}) + p_{\eta\eta\eta}\,(Q_{\eta} - Q_{\eta\eta\eta}) \tag{21}$$

The likes of such higher order terms do not appear in any known physical model of a dissipative fluid flow. It is highly questionable what the effects of these artificial terms might be on the existence and uniqueness of the solution of the Euler equations [13], since the extraneous higher order derivatives are not matched by any additional boundary conditions.

4. NAVIER STOKES EQUATIONS (NSE)

The NSE for unsteady, viscous, laminar flow allowing for heat conduction (assuming Fourier's law) expressed in non-dimensional form and non-orthogonal curvilinear coordinates can be summarized as

$$Q_t + E_\xi + F_\eta = \frac{1}{Re}\,(E^v_\xi + F^v_\eta) \tag{22}$$

where E^v_ξ, F^v_η incorporate physically dissipative terms due to shear viscosity, secondary viscosity and heat conductivity. The generalized viscous flux vectors are

$$E^v = \frac{1}{J}\begin{bmatrix} 0 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix} ; \; F^v = \frac{1}{J}\begin{bmatrix} 0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{23}$$

where

$$\begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix} = J\begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} E_3 \\ F_3 \end{bmatrix} = J\begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}\left(\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} q_x \\ q_y \end{bmatrix}\right) \tag{25}$$

Here, the components of the non-dimensional viscous stress tensor expressed in terms of ξ,η coordinates are:

$$\tau_{xx} = \mu^{*}\,[(y_\eta u_\xi - y_\xi u_\eta) + \lambda\,(x_\xi v_\eta - x_\eta v_\xi)]J \tag{26}$$

$$\tau_{yy} = \mu^{*}\,[(x_\xi v_\eta - x_\eta v_\xi) + \lambda(y_\eta u_\xi - y_\xi u_\eta)]J \; ; \; \tau_{xy} = \mu\,[x_\xi u_\eta - x_\eta u_\xi + y_\eta v_\xi - y_\xi v_\eta]J \tag{27}$$

and the non-dimensional heat conduction fluxes are

$$q_x = \frac{\mu}{(\gamma-1)M_\infty^2 Pr}\,(y_\eta T_\xi - y_\xi T_\eta)J \; ; \qquad q_y = \frac{\mu}{(\gamma-1)M_\infty^2 P_r}\,(x_\xi T_\eta - x_\eta T_\xi)J \tag{28}$$

Notice that Rankine-Hugoniot shock jump conditions are possible only if Stokes hypothesis ($\lambda/\mu = -2/3$) is enforced [10].

5. PHYSICALLY BASED DISSIPATION (PBD) FOR EULER AND NSE

Instead of intentionally using the non-physical formulations, the artificial dissipation can be modeled so that it is physically consistent.

To solve the Euler equations of inviscid flow, one could solve the complete NSE while enforcing perfect slip boundary conditions and spatially varying coefficients of viscosity [5,13]. This PBD model is physically consistent, since Euler equations represent an extreme case of NSE when the physical dissipation becomes negligible.

When solving the actual NSE, one can enforce no-slip solid body boundary conditions while augmenting the viscosity ceofficients in a same manner as it is often performed in a simplistic turbulence modeling. For example, the modified coefficient of viscosity can be of the form.

$$\mu_{PBD} = \mu\,(A + \nu) \qquad (29)$$

where ν is the artificial dissipation sensor [2,14], while constant A is zero for Euler equations and one for NSE. The "sensor" [14] can be based on first derivative of pressure or Mach number, second derivative of pressure (Eq. 20), vorticity (Fig. 3), divergence (Fig. 4) of the velocity vector, entropy, etc. It was shown [14] that different types of sensors can significantly affect the results and the convergence rates of both artificial dissipation [12] and the PBD [5,2] formulations. Nevertheless, the PBD formulation guarantees non-negative dissipation since it uses actual Navier-Stokes dissipative terms with altered coefficients of viscosity. Also, PBD does not alter mass conservation and it does not need higher order terms except those that exist in the NSE. The PBD concept for both Euler and NSE proves to be highly reliable [5,13,2,14], easy to comprehend and implement in the existing algorithms.

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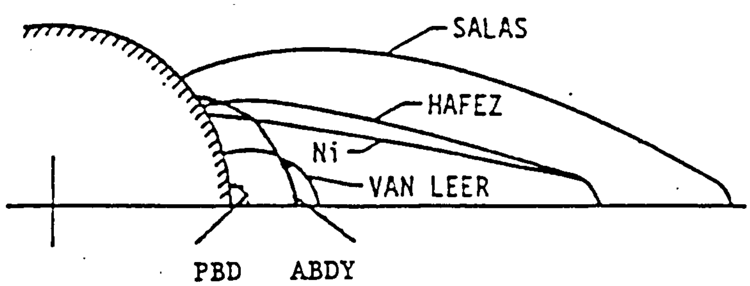

Fig. 1 Inviscid shock-induced separation: Euler equations and flow around a circle at $M_\infty = 0.5$ [4].

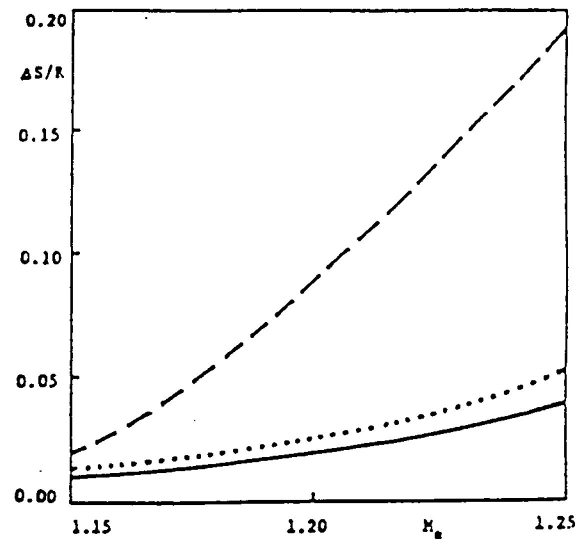

Fig. 2 Normal shock entropy rise ($\Delta S/R$) at different upstream Mach numbers: (——) PDP; (----) AMF; (– – –) ADV [10].

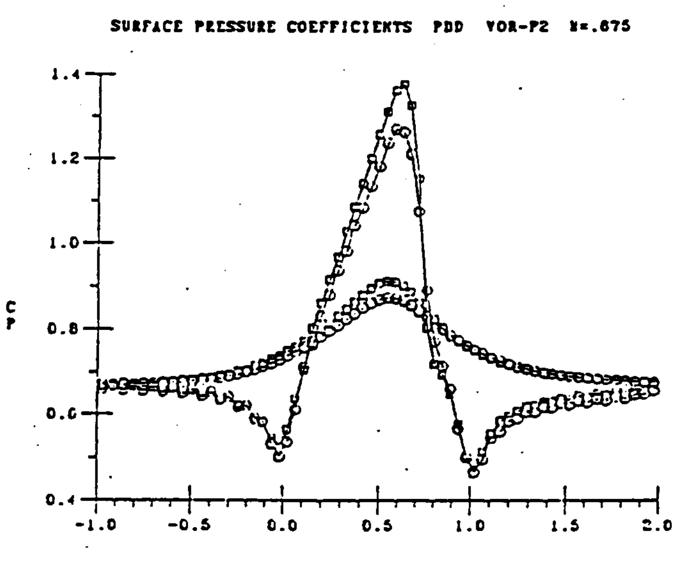

Fig. 3 Shocked channel flow: Euler equations with PBD and (ooo) Eq. 20 as a sensor; (☐☐☐) with $\nabla \times V$ as a sensor [14]

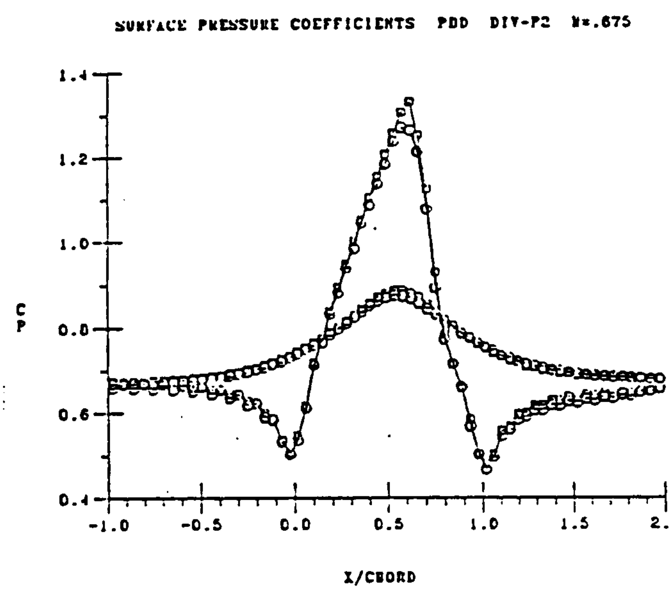

Fig. 4 Shocked channel flow: Euler equations with PBD and (ooo) Eq. 20 as a sensor; (☐☐☐) with $\nabla \bullet V$ as a sensor [14]