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ABSTRACT

A new approach to the inverse design of two-dimensional aerodynamic shapes has been developed. This formulation is based on a Stream-Function-Coordinate (SFC) concept for steady, irrotational, compressible, inviscid, planar flows. It differs from the classical stream function formulation in that it treats the y-coordinate of each point on a streamline as a function of the x-coordinate and the stream function ψ, that is, $Y = Y(x,\psi)$. This new formulation is especially suitable for the computation of stream line shapes, and therefore, for determination of aerodynamic shapes subject to specified surface pressure distributions. An additional advantage of this new formulation is that it requires the generation of only a one-dimensional grid in the x-direction. The grid in the y-direction is computed as a part of the solution since y-coordinates of the streamlines are treated as the unknowns in the SFC formulation. In addition, the SFC method is equally suitable for the analysis of the flowfields around given shapes. A computer code has been developed on the basis of SFC formulation. It is capable of performing flowfield analysis and inverse design of airfoil cascade shapes by changing a single input parameter.

INTRODUCTION

In a recently published article, Huang and Dulikravich[1] gave detailed derivations of the new Stream Function Coordinate (SFC) concept for inviscid, steady, two-dimensional and three-dimensional compressible flows. The SFC concept reflects the main objective of the inverse design where the ultimate goal is to determine the shape, that is, the coordinates of a surface contour which is compatible with the desired surface pressure distribution. Thus, it is logical to solve for the coordinates directly. Recently, Chen and Zhang[2] have published a paper on inverse design of multiple cascade shapes. They used a special form of the SFC formulation suitable for axisymmetric surfaces of turbomachinery and they have successfully computed shapes of simple cascades as well as shapes of multiple cascades with splitter blades inside the flow passages. Oven and Pearson[3] have developed a complete threedimensional formulation based on a general concept by solving directly for the coordinates. They have applied their formulation to different duct flows and to free jet flows[4].

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ANALYSIS

Instead of using the standard formulation where the stream function ψ is a function of the x and y coordinates

$$(1 - K^2\psi_y^2)\,\psi_{xx} + 2K^2\,\psi_x\psi_y\,\psi_{xy} + (1 - K^2\psi_x^2)\,\psi_{yy} = 0 \qquad (1)$$

Huang and Dulikravich [1] performed a transformation

$$\psi = \psi(x,y) \;\text{--}\; Y = Y(x,\psi) \qquad (2)$$

which transforms (1) into the SFC equation

$$(Y_\psi^2 - K^2)\,Y_{xx} - 2\,Y_x Y_\psi Y_{x\psi} + (1 + Y_x^2)\,Y_{\psi\psi} = 0 \qquad (3)$$

where the Y-coordinate of each streamline is treated as an unknown and x and ψ are known. Here, the compressibility coefficient K^2 is defined as

$$K^2 = \frac{1}{\left(\frac{\rho}{\rho_*}\right)^2 \left(\frac{a}{a_*}\right)^2} \qquad (4)$$

where ρ is the local density and a is the local speed of sound. Details of the derivation and evaluation of K^2 are given in Appendix A. The SFC formulation has significant advantages over the classical stream function formulation where $\psi = \psi(x,y)$. For two-dimensional problems SFC requires only a one-dimensional grid in the x-direction. The other family of grid lines is determined as a part of the solution where Y are the unknown coordinates of the streamlines ψ = constant. Because of the SFC formualation, true upwind differencing could be achieved without the complexity of determining the direction of the local velocity vectors since one family of the grid lines corresponds to the streamlines. This simplifies the extension of the code to transonic flows[2]. Huang and Dulikravich[1] clearly pointed out that the SFC formulation where $Y = Y(x,\psi)$ is singular at all locations where the x-component of the velocity vector becomes zero. These singularities are nonphysical since they are created by the transformation and cannot be eliminated simply by using grid clustering within the regions of singular points[1]. Thus, strictly speaking, the SFC formulation is suitable for the flow field analysis and shape inverse design of objects having cusped leading and trailing edge points where there are no stagnation points. In practice, leading and trailing edges are often[5] modified when using the SFC formulation by adding artificial cusps to them.

8. Ives, D., "Approximate Inversion of the Stream-Tube-Area Relations," Grumman Research Dept. Memo., RM-452, July 1969.

APPENDIX A: Derivation of the Compressibility Coefficient K

Notice that

$$\frac{T}{T_o} = \left(\frac{2}{\gamma + 1}\right) \frac{T}{T_*} \tag{A.1}$$

where T is the absolute temperature, T_o is stagnation temperature and T_* is the critical temperature. In terms of local Mach numbers this becomes

$$\frac{1}{\left[1 + \frac{\gamma - 1}{2} M^2\right]} = \left(\frac{2}{\gamma + 1}\right) . \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right] \tag{A.2}$$

This can be rewritten as

$$\frac{\gamma + 1}{2} \left(\frac{1}{\left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]^{\frac{\gamma + 1}{\gamma - 1}}}\right)^{\frac{\gamma - 1}{\gamma + 1}} = \left[1 + \frac{\gamma - 1}{2} M^2\right] \tag{A.3}$$

Nevertheless

$$M^2 = \frac{M_*^2}{a^2/a_*^2} = \frac{M_*^2}{\left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]} \tag{A.4}$$

Hence, A.3 can be rewritten as

$$\frac{\gamma + 1}{2} \left(\frac{M^2}{M_*^2} \frac{1}{\left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]^{\frac{2}{\gamma - 1}}}\right)^{\frac{\gamma - 1}{\gamma + 1}} - \frac{\gamma - 1}{2} M^2 - 1 = 0 \tag{A.5}$$

Since

$$\frac{\rho}{\rho_*} = \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]^{\frac{1}{\gamma - 1}} \tag{A.6}$$

it follows that A.5 becomes

$$\frac{\gamma + 1}{2} \left(\frac{M^2}{\left(\frac{\rho}{\rho_*}\right)^2 M_*^2}\right)^{\frac{\gamma - 1}{\gamma + 1}} - \frac{\gamma - 1}{2} M^2 - 1 = 0 \tag{A.7}$$

Notice now that

$$M^2 = \left(\frac{\rho}{\rho_*}\right)^2 M_*^2 \left(\frac{1}{\left(\frac{\rho}{\rho_*}\right)^2 \left(\frac{a}{a_*}\right)^2}\right) = \left(\frac{\rho}{\rho_*}\right)^2 M_*^2 K^2 \tag{A.8}$$

Since the compressibility coefficient K is defined as

$$K^2 = \frac{1}{\left(\frac{\rho}{\rho_*}\right)^2 \left(\frac{a}{a_*}\right)^2} = \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]^{-\left(\frac{\gamma + 1}{\gamma - 1}\right)} \tag{A.9}$$

it follows that A.7 divided with $(\gamma-1)$ becomes

$$\frac{\gamma + 1}{\gamma - 1} (K^2)^{\frac{\gamma - 1}{\gamma + 1}} - \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_*^2\right]^2 M_*^2 K^2 - \frac{2}{\gamma - 1} = 0 \tag{A.10}$$

At the same time notice that

$$\left(\frac{\rho}{\rho_*}\right)^2 M_*^2 = \left(\frac{\rho}{\rho_*} \frac{u}{a_*}\right)^2 + \left(\frac{\rho}{\rho_*} \frac{v}{a_*}\right)^2 = (\psi_y)^2 + (\psi_x)^2 = \frac{1}{Y_\psi^2} + \left(\frac{Y_x}{Y_\psi}\right)^2 = \frac{1 + Y_x^2}{Y_\psi^2} \tag{A.11}$$

Finally, A.10 becomes

$$\frac{\gamma + 1}{\gamma - 1} (K^2)^{\frac{\gamma - 1}{\gamma + 1}} - \left(\frac{1 + Y_x^2}{Y_\psi^2}\right) K^2 - \frac{2}{\gamma - 1} = 0 \tag{A.12}$$

Since Y_x and Y_ψ will be changing during the iterative process, this means that

$$F = \frac{\gamma + 1}{\gamma - 1} (K^2)^{\frac{\gamma - 1}{\gamma + 1}} - \left(\frac{1 + Y_x^2}{Y_\psi^2}\right) K^2 - \frac{2}{\gamma - 1} \neq 0 \tag{A.13}$$

Thus, at every point in the flow field for the given instantaneous values of Y_x and Y_ψ we can iteratively determine the corresponding instantaneous local values of the compressibility coefficient K. Second order (modified) Newton's iteration yields

$$\frac{dF}{dK} = 2 \left[K^{\frac{\gamma - 3}{\gamma + 1}} - \left(\frac{1 + Y_x^2}{Y_\psi^2}\right) K\right] \tag{A.14}$$

$$\frac{d^2F}{dK^2} = 2 \left[\frac{\gamma - 3}{\gamma + 1} K^{\frac{-4}{\gamma + 1}} - \left(\frac{1 + Y_x^2}{Y_\psi^2}\right)\right] \tag{A.15}$$

so that

$$K^{(n+1)} = K^{(n)} - F \frac{dF}{dK} \Big/ \left[\left(\frac{dF}{dK}\right)^2 - F \frac{d^2F}{dK^2}\right] \tag{A.16}$$

where the superscript n is the iteration counter.

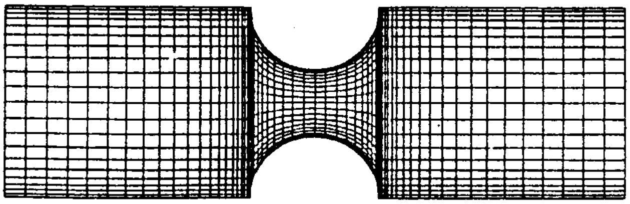

Fig. 1 Initial Y-x grid consisting of (20+20+20) x 20 cells for the flow through a cascade of dipoles

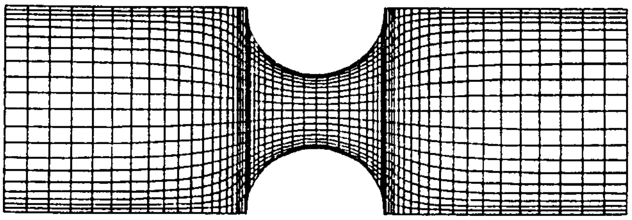

Fig. 2 Final streamline shapes for the flow through a cascade of dipoles

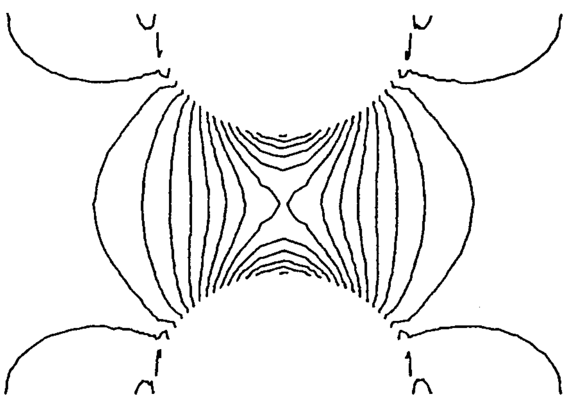

Fig. 3 Computed isobars for the $M_\infty = 0.05$ flow through a cascade of dipoles

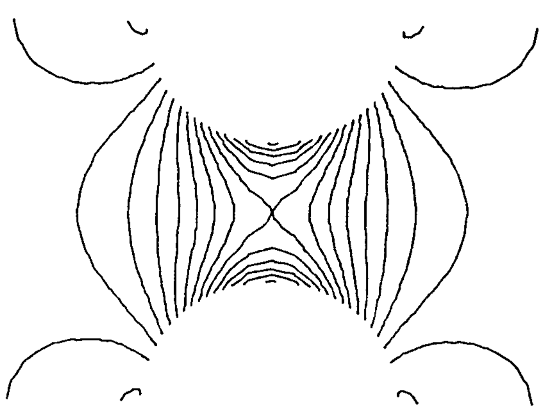

Fig. 4 Analytic values for isobars for an incompressible flow through a cascade of dipoles

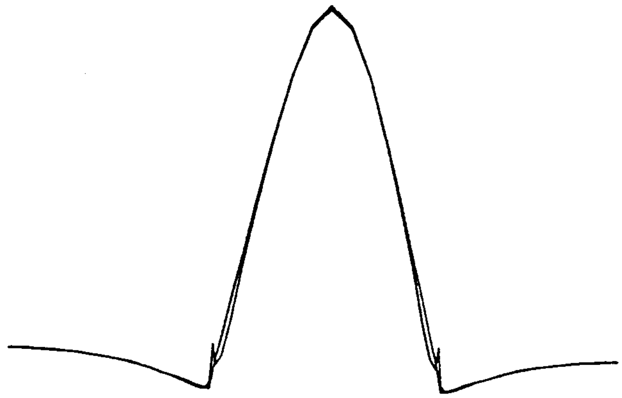

Fig. 5 Superimposed analytic and computed surface C_p values for the $M_\infty = 0.0$ and $M_\infty = 0.05$ flow through a cascade of dipoles

Fig. 6 Intermediate shapes of the bottom wall during the inverse design from a cascade of dipoles to a straight channel

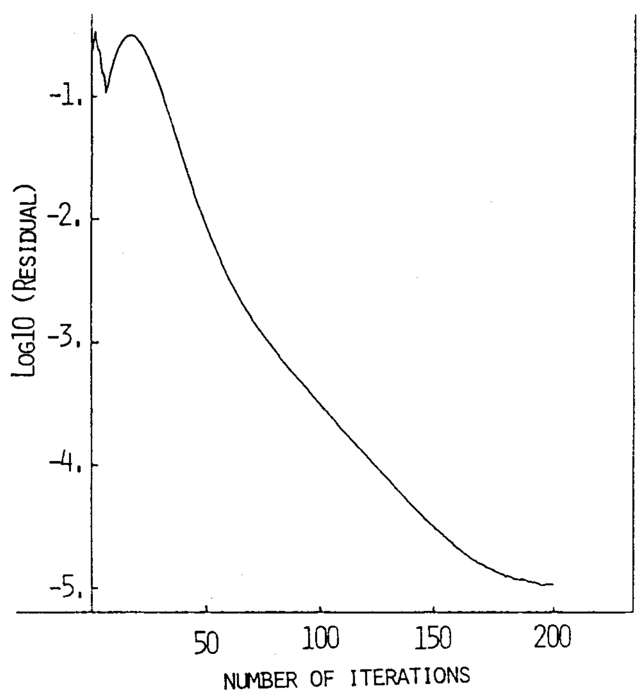

Fig. 7 Convergence history for the inverse design from a cascade of dipoles to a straight channel