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A CAVITATION MODEL BASED ON GASDYNAMIC FORMULATION

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ABSTRACT

A new analytic model has been developed that governs steady flow of inviscid compressible homogeneous mixture of liquid and vapor generated by pure flashing. The derived hydrodynamic and thermodynamic relations involve the effects of latent heat release during evaporation/condensation without heat transfer. Specifically, it shows how the latent heat release affects the local speed of sound of a bubbly mixture.

Nomenclature:

a = local speed of sound in the mixture

L = latent heat of evaporation (condensation) per unit mass of liquid

M = mass

p = thermodynamic pressure of the mixture

Q = heat (per unit total mass) transfered to (from) the mixture

R = specific gas constant

T = absolute temperature of the mixture

V = volume

W = reversible work (per unit total mass) performed on the mixture

δ = volume ratio (vapor/liquid)

μ = mass ratio (vapor/liquid)

ρ = density

INTRODUCTION

Computational methods for the prediction (analysis of complex threedimensional flowfields involving homocompositional fluids have been developed and are widely used in aerospace industry. It has been only recently that similar efforts have been initiated in the hydrodynamics. The main physical phenomena in hydrodynamics is cavitation [1]. Since it is still inadequately understood, the existing mathematical models for cavitating (flashing) liquids are not general enough. It would be very desirable to model a liquid-vapor mixture using the same general conservation laws appropriate for gases [2,3]. Then, we could use the exiting sophisticated large computer programs from gasdynamics and apply them directly in the field of hydraulic turbomachinery design.

ANALYSIS

Ratio of the vapor mass and the liquid mass in a two-component homogeneous cavitating mixture is

$$\mu = \frac{M_v}{M_\ell} \tag{1}$$

Average mixture density, ρ, is then defined as

$$\frac{M_v}{\rho_v} + \frac{M_\ell}{\rho_\ell} = \frac{M_v + M_\ell}{\rho} \tag{2}$$

Volume ratio of the vapor and liquid phases is given as

$$\delta = \frac{V_v}{V_\ell} = \frac{M_v \rho_\ell}{M_\ell \rho_v} = \frac{\rho_v V_v \rho_\ell}{\rho_\ell V_\ell \rho_v} = \mu \frac{\rho_\ell}{\rho_v} \tag{3}$$

Then, from Eq. 2

$$\frac{1}{\rho_v} \frac{M_v}{M_\ell} + \frac{1}{\rho_\ell} = \frac{M_v + M_\ell}{\rho M_\ell} = \frac{1 + \mu}{\rho} = \frac{\mu}{\rho_v} + \frac{1}{\rho_\ell} \tag{4}$$

From Eq. 3 and Eq. 4, it follows that

$$\frac{\delta}{\rho_\ell} + \frac{1}{\rho_\ell} = \frac{1+\mu}{\rho} \quad \therefore \quad \frac{1+\delta}{\rho_\ell} = \frac{1+\mu}{\rho} \tag{5}$$

Hence

$$\frac{\rho}{\rho_\ell} = \frac{1+\mu}{1+\delta} \tag{6}$$

and

$$\frac{1}{\rho_v} = \frac{1}{\mu} \left(\frac{1+\mu}{\rho} - \frac{1}{\rho_\ell}\right) \tag{7}$$

Conservation of energy for the vapor-liquid mixture is given [14] as

$$(M_v + M_\ell)\, dQ = (M_v C_v + M_\ell C_\ell)\, dT + (M_v + M_\ell)\, dW - M_\ell\, L \tag{8}$$

Hence

$$dQ = \left(\frac{\mu C_v + C_\ell}{1 + \mu} \right) dT + dW - \left(\frac{1}{1 + \mu} \right) L \tag{9}$$

But

$$dW = p\, d\left(\frac{1}{\rho}\right) = p \left(- \frac{d\rho}{\rho}\right) = - \frac{p}{\rho^2}\, d\rho \tag{10}$$

Hence, Eq. 9 becomes

$$\frac{p}{\rho^2}\, d\rho = \left(\frac{\mu\, C_v + C_\ell}{1 + \mu}\right) dT - dQ - \left(\frac{1}{1 + \mu}\right) L \tag{11}$$

Equation of state for the vapor component can be approximated as

$$\frac{p}{\rho_v} = RT \tag{12}$$

From Eq. 7, it follows that

$$\frac{p}{\mu} \left(\frac{1 + \mu}{\rho} - \frac{1}{\rho_\ell}\right) = RT \tag{13}$$

After differentiating Eq. 13, the result is

$$\frac{dp}{\mu} \left(\frac{1 + \mu}{\rho} - \frac{1}{\rho_\ell}\right) + \frac{p}{\mu}\left(- \frac{1 + \mu}{\rho^2}\, d\rho\right) = R\, dT \tag{14}$$

From Eq. 11, it follows that

$$dT = \left(\frac{1 + \mu}{\mu\, C_v + C_\ell}\right) \frac{p\, d\rho}{\rho^2} + \left(\frac{1 + \mu}{\mu\, C_v + C_\ell}\right) dQ + \left(\frac{1}{\mu\, C_v + C_\ell}\right) dL \tag{15}$$

From Eq. 14, it follows that

$$\frac{dp}{p} = \frac{\mu\, R\, d\, T}{p \left(\frac{1 + \mu}{\rho} - \frac{1}{\rho_\ell}\right)} + \frac{\left(\frac{1}{1 + \mu}\right)^{-1} \frac{d\rho}{\rho^2}}{\left(\frac{1 + \mu}{\rho} - \frac{1}{\rho_\ell}\right)} \tag{16}$$

Introducing Eq. 15 into Eq. 16 results in

$$\frac{dp}{p} = \frac{(1 + \mu)\, \mu\, R}{(\mu\, C_v + C_\ell)\left((1 + \mu) - \frac{\rho}{\rho_\ell}\right)} \frac{d\rho}{\rho} + \frac{(1 + \mu)}{\left((1 + \mu) - \frac{\rho}{\rho_\ell}\right)} \frac{d\rho}{\rho}$$

$$+\frac{(1+\mu)\ \mu\ R\ \rho\ dQ}{p(\mu\ C_v + C_\ell)\ ((1+\mu) - \frac{\rho}{\rho_\ell})} + \frac{\mu\ R\ \rho\ \ dL}{p(\mu\ C_v + C_\ell)\ ((1+\mu) - \frac{\rho}{\rho_\ell})} \qquad (17)$$

Notice that the gas constant is

$$R = C_p - C_v \qquad (18)$$

and that

$$\frac{\mu\ R\ \rho}{p} = \frac{\mu}{\rho_v}\ \frac{\rho}{T} = \frac{1}{T}\ ((1+\mu) - \frac{\rho}{\rho_\ell}) \qquad (19)$$

Let

$$\Gamma = \frac{\mu\ C_p + C_\ell}{\mu\ C_v + C_\ell} \qquad (20)$$

and

$$\Gamma_1 = \frac{dL/dT}{\mu\ C_v + C_\ell} \qquad (21)$$

If heat transfer is neglected (adiabatic process), then dQ=0 and Eq. 17 becomes

$$\frac{dp}{p} = \frac{\Gamma}{(1 - \frac{1}{1+\mu}\ \frac{\rho}{\rho_\ell})}\ \frac{d\rho}{\rho} + \Gamma_1\ \frac{dT}{T} \qquad (22)$$

Since (dL/dT) $\cong$ const. for, say, water [1], after integrating Eq. 22 from p_o to p_1, from ρ_o to ρ, and from T_o to T, the result is

$$\ell n\ p - \ell n\ p_o = -\Gamma \ell n\ (\frac{1}{\rho} - \frac{1}{1+\mu}\ \frac{1}{\rho_\ell}) \quad + \Gamma \ell n\ (\frac{1}{\rho_o} - \frac{1}{1+\mu}\ \frac{1}{\rho_\ell})$$

$$+ \Gamma_1 \ell n T \qquad - \Gamma_1 \ell n\ T_o \qquad (23)$$

Hence

$$p\ (\frac{1}{\rho} - \frac{1}{1+\mu}\ \frac{1}{\rho_\ell})^{\Gamma} = T^{\Gamma_1} C_1 \qquad (24)$$

where

$$C_1 = p_o \left(\frac{1}{\rho_o} - \frac{1}{1+\mu}\frac{1}{\rho_\ell}\right)^{\Gamma} T_o^{-\Gamma_1} \tag{25}$$

Then Eq. 24 gives

$$p \left(\frac{\mu}{(1+\mu)\,\rho_v}\right)^{\Gamma} T^{-\Gamma_1} = p_o \left(\frac{\mu}{(1+\mu)\rho_{vo}}\right)^{\Gamma} T_o^{-\Gamma_1} \tag{26}$$

From equation of state (Eq. 12) it follows that

$$p \left(\frac{\mu}{1+\mu}\frac{RT}{p}\right)^{\Gamma} T^{-\Gamma_1} = p_o \left(\frac{\mu}{1+\mu}\frac{RT_o}{p_o}\right)^{\Gamma} T_o^{-\Gamma_1} = C_1 = \text{const.} \tag{27}$$

Hence

$$p^{1-\Gamma}\, T^{\Gamma-\Gamma_1} \left(\frac{\mu R}{1+\mu}\right)^{\Gamma} = p_o^{1-\Gamma}\, T_o^{\Gamma-\Gamma_1} \left(\frac{\mu R}{1+\mu}\right)^{\Gamma} \tag{28}$$

After exponentiating both sides to $(-\frac{1}{\Gamma})$ it follows that

$$\frac{p^{\frac{\Gamma-1}{\Gamma}}}{T^{1-\Gamma_2}} = \frac{p_o^{\frac{\Gamma-1}{\Gamma}}}{T_o^{1-\Gamma_2}} = C_2 = \text{const.} \tag{29}$$

where

$$\Gamma_2 = \frac{dL/dT}{\Gamma(\mu\, C_v + C_\ell)} = \frac{dL/dT}{\mu\, C_p + C_\ell} = \frac{\Gamma_1}{\Gamma} \tag{30}$$

For a pure vapor ($\mu \to \infty$) it follows that $\Gamma \to \gamma = C_p/C_v$ and $\Gamma_2 \to 0$.

Consequently, for a pure vapor

$$\frac{p^{\frac{\gamma-1}{\gamma}}}{T} = \frac{p_o^{\frac{\gamma-1}{\gamma}}}{T_o} = \text{const.} \tag{31}$$

This is the relation for polytropic processes in ideal gases. When the vapor does not exist ($\mu \to 0$), it follows that $\Gamma \to 1$ and $\Gamma_2 \to (dL/dT)/C$.

Hence

$$\frac{p^{\frac{1-1}{1}}}{T^{1-\Gamma_2}} = \frac{p_o^{\frac{1-1}{1}}}{T_o^{1-\Gamma_2}} = \text{const.} \tag{32}$$

This is possible only if $T = T_o$ = const. (isothermal liquid).

LOCAL SPEED OF SOUND

The Euler equation for steady, one-dimensional momentum conservation is

$$u\, du = -\frac{dp}{\rho} \tag{33}$$

where u is the local speed of the mixture. This equation can be integrated if ρ is extracted from Eq. 24 and T is expressed in terms of ρ by using Eq. 12 and Eq. 7.

$$T = \frac{p}{R\rho_v} = \frac{p}{R\mu}\left(\frac{1+\mu}{\rho} - \frac{1}{\rho_\ell}\right) \tag{34}$$

Then, from Eq. 24

$$p\left(\frac{1}{\rho} - \frac{1}{1+\mu}\frac{1}{\rho_\ell}\right)^{\Gamma} = \left(\frac{1+\mu}{R\mu}\right)^{\Gamma_1} p^{\Gamma_1}\left(\frac{1}{\rho} - \frac{1}{1+\mu}\frac{1}{\rho_\ell}\right)^{\Gamma_1} C_1 \tag{35}$$

Hence

$$\left(\frac{1}{\rho} - \frac{1}{1+\mu}\frac{1}{\rho_\ell}\right)^{\Gamma-\Gamma_1} = \left(\frac{1+\mu}{\mu R}\right)^{\Gamma_1} p^{\Gamma_1 - 1}\, C_1 \tag{36}$$

Consequently

$$\frac{1}{\rho} = \left(\frac{1}{1+\mu}\right)\frac{1}{\rho_\ell} + \left(\frac{1+\mu}{\mu R}\right)^{\frac{\Gamma_1}{\Gamma-\Gamma_1}} p^{\frac{\Gamma_1-1}{\Gamma-\Gamma_1}}\, C_1^{\frac{1}{\Gamma-\Gamma_1}} \tag{37}$$

Then, Eq. 33 after integration becomes

$$\frac{u^2}{2} - \frac{u_o^2}{2} = \frac{(p_o - p)}{\rho_\ell(1+\mu)} + \left(\frac{\mu R}{1+\mu}\right)^{\frac{-\Gamma_1}{\Gamma-\Gamma_1}} C_1^{\frac{1}{\Gamma-\Gamma_1}}\left(\frac{\Gamma-\Gamma_1}{\Gamma-1}\right)\left(p_o^{\frac{\Gamma_1-1}{\Gamma-\Gamma_1}+1} - p^{\frac{\Gamma_1-1}{\Gamma-\Gamma_1}+1}\right) \tag{38}$$

This can be written as

$$u^2 = u_o^2 + \frac{2(p_o - p)}{\rho_\ell\,(1+\mu)} + 2\left(\frac{\mu R}{1+\mu}\right)^{-\frac{\Gamma_1}{\Gamma-\Gamma_1}} C_1^{\frac{1}{\Gamma-\Gamma_1}}\left(\frac{\Gamma-\Gamma_1}{\Gamma-1}\right) p_o^{\frac{\Gamma-1}{\Gamma-\Gamma_1}}\left(1 - \left(\frac{p}{p_o}\right)^{\frac{\Gamma-1}{\Gamma-\Gamma_1}}\right) \tag{39}$$

But, from Eq. 27 it follows that

$$\left(\frac{C_1}{p_o}\right)^{\frac{1}{\Gamma-\Gamma_1}} = T_o^{\frac{\Gamma_1}{\Gamma-\Gamma_1}}\left(\frac{\mu R}{1+\mu}\frac{T_o}{p_o}\right)^{\frac{\Gamma}{\Gamma-\Gamma_1}} = \left(\frac{\mu}{1+\mu}\frac{1}{\rho_{vo}}\right)^{\frac{\Gamma}{\Gamma-\Gamma_1}} T_o^{\frac{\Gamma}{\Gamma-\Gamma_1}} \tag{40}$$

Hence, Eq. 39 becomes

$$u^2 = u_o^2 + \frac{2(p_o - p)}{\rho_\ell (1+\mu)} + p_o^{\frac{\Gamma-\Gamma_1+\Gamma_1}{\Gamma-\Gamma_1}} \left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{\rho_{vo}}\right)^{\frac{\Gamma-\Gamma_1+\Gamma_1}{\Gamma-\Gamma_1}} \left(\frac{1}{R}\right)^{\frac{\Gamma_1}{\Gamma-\Gamma_1}}$$

$$\left(\frac{1}{T_o}\right)^{\frac{\Gamma_1}{\Gamma-\Gamma_1}} \; 2 \left(\frac{\Gamma-\Gamma_1}{\Gamma-1}\right) \left(1 - \left(\frac{p}{p_o}\right)^{\frac{\Gamma-1}{\Gamma-\Gamma_1}}\right) \tag{41}$$

Finally

$$u^2 = u_o^2 + \frac{2\, p_o}{\rho_\ell (1+\mu)} \left\{\left(\frac{\Gamma-\Gamma_1}{\Gamma-1}\right) \delta_o \left(1 - \left(\frac{p}{p_o}\right)^{\frac{\Gamma-1}{\Gamma-\Gamma_1}}\right) + \left(1 - \frac{p}{p_o}\right)\right\} \tag{42}$$

where

$$\delta_o = \mu \frac{\rho_\ell}{\rho_{vo}} \tag{43}$$

From Eq. 22, the local speed of sound, a, is obtained as

$$a^2 = \frac{dp}{d\rho} = \frac{\Gamma}{\left(1 - \frac{1}{1+\mu}\frac{\rho}{\rho_\ell}\right)} \frac{p}{\rho} + \frac{dL/dT}{(\mu C_v + C_\ell)} \frac{p}{T} \frac{dT}{d\rho} \tag{44}$$

Also, notice that [2]

$$\frac{dp}{d\rho} = \frac{L}{T} \frac{1}{\left(1 - \frac{1}{1+\mu}\frac{\rho}{\rho_\ell}\right)} \frac{dT}{d\rho} \tag{45}$$

Hence

$$a^2 = \left(\frac{\Gamma p}{\rho_v}\right) \frac{\rho_v}{\left(\frac{1+\mu}{\rho} - \frac{1}{\rho_\ell}\right)} \frac{1}{\frac{\rho^2}{1+\mu}} + \frac{p}{L} \frac{dL/dT}{(\mu C_p + C_\ell)} \left(\frac{1}{\rho_v} - \frac{1}{\rho_\ell}\right) \frac{dp}{d\rho} \tag{46}$$

From Eq. 4 it follows that

$$\left(1 - \frac{\rho v}{\rho_\ell}\right) = \left(1 - \frac{\rho v}{\rho}\right)(1+\mu) \tag{47}$$

Using Eq. 7, we get

$$a^2 = (\frac{\Gamma p}{\rho_v}) \frac{\rho_v^2}{\rho^2} (\frac{1 + \mu}{\mu}) / \{(\frac{-dL/dT}{\mu C_v + C_\ell}) \frac{p}{\rho_v L} (1 + \mu)(1 - \frac{\rho_v}{\rho}) + 1\} \qquad (48)$$

Notice that

$$\frac{\rho_v^2}{\rho^2} (\frac{1 + \mu}{\mu}) = \frac{\frac{\mu^2}{\delta^2} \rho_\ell^2}{\rho^2} (\frac{1 + \mu}{\mu}) = \frac{\mu}{\delta^2} (1 + \mu) \frac{(1 + \delta)^2}{(1 + \mu)^2} \qquad (49)$$

Finally, the local speed of sound of a liquid-vapor mixture is

$$a^2 = (\frac{\Gamma p}{\rho_v})\{(\frac{\mu}{1 + \mu})(1 + \frac{1}{\delta})^2 / \{1 - (\frac{\Gamma_1 p}{\rho_v}) \frac{1}{L} (1 - \frac{\mu}{\delta})\} \qquad (50)$$

Since Γ_1 is proportional to (dL/dT) and since (dL/dT) is negative [1] it follows from Eq. 50 that the latent heat of evaporation reduces the local speed of sound.

REFERENCES:

1. Isay, W. H., "Kavitation", Schiffahrts-Verlag "Hansa", C. Schroedter & Co., Hamburg, 1984.

2. Isay, W. H. and Roestel, T., "Druckverteilung an Flugelprofilen in gashaltiger Wasserstroemung," ZAMM, 1975, pp. 580-588.

3. Savu, G., "The Cloud Nozzle," XIV Int. Symp. on Rarefied Gasdynamics, Tsukuba Science City, Japan, July 16-20, 1984.

4. Tangren, R. F., Dodge, C. H. and Seifert, H. S., "Compressibility Effects in Two-Phase Flows," Journal of Applied Physics, Vol. 20, No. 7, July 1949, pp. 637-645.