Analysis of Artificial Dissipation Models for the Transonic Full-Potential Equation

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Artificial density or viscosity (ADV) and artificial mass flux (AMF) concepts used in the iterative algorithms for the numerical solution of the transonic full potential equation (FPE) have been analyzed and compared with the exact physically dissipative potential (PDP) flow equation. Coefficients of the derivatives in the existing artificial dissipation models were found to produce only several of the physically existing derivatives. Moreover, the common artificial density and artificial viscosity formulations generate terms of the wrong magnitude, and even of the wrong sign, when compared to the PDP formulation. The AMF formulation, although imperfect, is shown to offer an alternative to the artificial density and artificial viscosity concepts.

Introduction

COMPUTATIONAL fluid dynamics of transonic flows were based for a number of years on the transonic full potential equation (FPE) as a viable mathematical model. The iterative algorithms capable of capturing isentropic discontinuities in the solution of the artificially time-dependent[1] and artificially dissipative[2-4] FPE became a standard aerodynamic analysis and design tool.

In addition, type-dependent rotated finite differencing[2] is usually employed to numerically mimic the locally proper analytic domain of dependence of the governing partial differential equation. This means that the second derivative of potential function ϕ in the streamline direction s should be evaluated using upstream differentiation only when the FPE is locally hyperbolic ($M^2 > 1$). Consequently, only coalescence of a preferred family of characteristics (compression waves) is allowed to occur, resulting in acceptable isentropic discontinuities (compression shocks). Expansion shocks, which are impossible for calorically perfect gases, should be thus avoided. Explicit artificial dissipation of the artificial density[4] or artificial viscosity[2,3] type is added in a fully conservative form[3] in an attempt to nullify the truncation errors introduced when using upstream differentiation locally in supersonic regions of the flowfield. The similarity of artificial density and its truncated equivalent, called artificial viscosity, is outlined in Appendix A.

Numerical solutions of the multidimensional FPE with the artificial viscosity[2,3] or the artificial density[4] frequently exhibit spurious oscillations behind the shock (Fig. 1) and sudden overshoots ahead of the shock.[5] Although the computed pressures on the surfaces of the objects are seemingly correct, the computed shocks diffuse quickly with the growing distance from the boundary (Fig. 2). These numerically generated phenomena can be observed on a fine grid when the entire field of isobars is plotted.

A number of different analytic formulations[6-8] for the artificial dissipation were developed in the past. Nevertheless, there were only a few isolated attempts[9,10] at analytically analyzing these concepts and suggesting possible reasons for the numerically obtained results.

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The objective of this paper is to expose clearly all the terms generated by the artificial density[4] or viscosity[2,3] (ADV) schemes, the artificial mass flux (AMF)[9,10] scheme, and the directional flux biasing (DFB)[11-13] formulation, and to relate them to the terms that exist in a PDP flow equation.[14]

The Full-Potential Equation

Mass conservation for steady homentropic and homenergetic irrotational flows of inviscid fluids without body forces and without mass sources or sinks is given as

$$\nabla \cdot (\rho V) = \nabla \cdot (\rho \nabla \phi) = 0 \tag{1}$$

where ρ is the local fluid density and V the local velocity vector. For the sake of simplicity, further analysis will be performed in two dimensions. Expressed in a locally streamline-aligned (s,n) orthogonal, two-dimensional coordinate system, Eq. (1) becomes

$$\nabla \cdot (\rho V) = \left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial n}\hat{e}_n\right) \cdot \left(\rho \frac{\partial \phi}{\partial s}\hat{e}_s + \rho \frac{\partial \phi}{\partial n}\hat{e}_n\right) = 0 \tag{2}$$

where $\hat{e}_s$ and $\hat{e}_n$ are the unit vectors in s and n direction, respectively.

Let

$$\frac{\partial \phi}{\partial s} = \phi_s, \qquad \frac{\partial \phi}{\partial n} = \phi_n \tag{3}$$

$$V = \phi_s \hat{e}_s, \qquad \phi_n = 0 \tag{4}$$

Then

$$\nabla \cdot (\rho V) = \rho[\phi_{ss} + \phi_{nn} + (\rho_s/\rho)\phi_s] = 0 \tag{5}$$

The local speed of sound a, normalized with the critical speed of sound a_*, becomes

$$\frac{a^2}{a_*^2} = \left\{\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\left[\left(\frac{\phi_s}{a_*}\right)^2 + \left(\frac{\phi_n}{a_*}\right)^2\right]\right\} \tag{6}$$

where γ is the ratio of specific heats, and the gas is assumed to be calorically perfect. Similarly

$$\frac{\rho}{\rho_*} = \left\{\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\left[\left(\frac{\phi_s}{a_*}\right)^2 + \left(\frac{\phi_n}{a_*}\right)^2\right]\right\}^{\frac{1}{\gamma-1}} \tag{7}$$

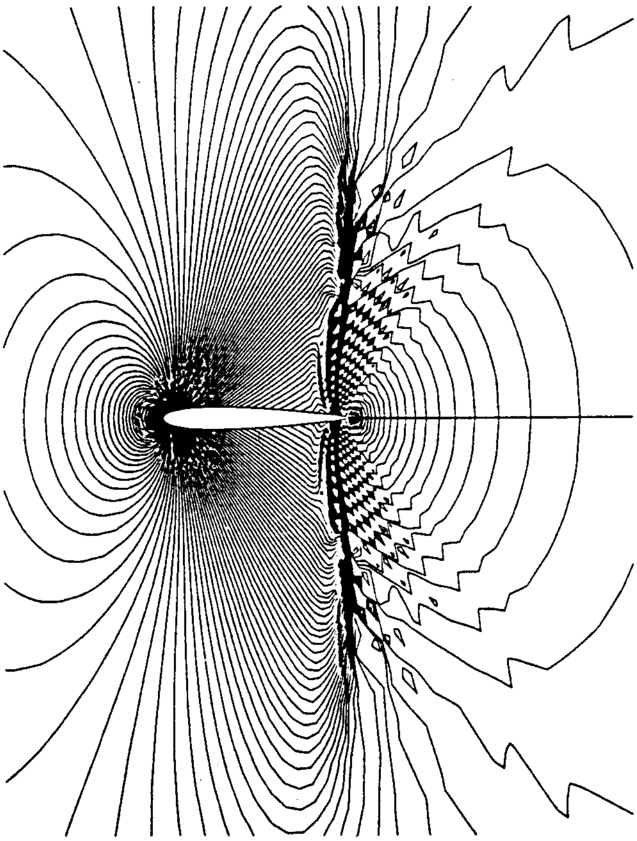

Fig. 1 Iso-Mach lines for a fine grid (256 × 48) solution of an FPE with ADV formulation. Airfoil is NACA 0012 with $M = 0.94$. Notice a "checkerboard" pattern along the shock.

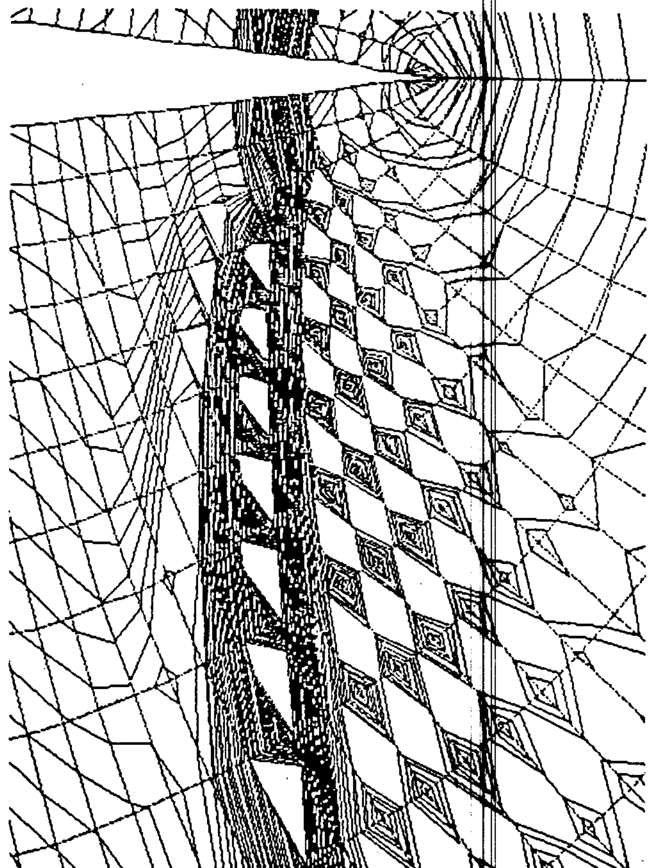

Fig. 2 Enlarged region of the shock wave showing that the surface pressures computed with ADV could quickly diffuse.

In order to simplify notation, let

$$a^2 = \frac{a^2}{a_*^2}, \qquad \rho = \frac{\rho}{\rho_*}$$

$$\phi_s = \frac{\phi_s}{a_*}, \qquad \phi_n = \frac{\phi_n}{a_*} = 0 \tag{8}$$

Then, from Eq. (7) it follows that

$$\frac{\rho_s}{\rho}\phi_s = \frac{\phi_s}{\rho}\left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2}(\phi_s)^2\right]^{\frac{1}{\gamma-1}-1}(-\phi_s\phi_{ss}) \tag{9}$$

since $\phi_n = 0$ by definition [Eq. (4)]. Hence, from Eq. (9) and Eq. (6) it follows that

$$\frac{\rho_s}{\rho}\phi_s = -\frac{\phi_s^2}{a^2}\phi_{ss} = -M^2\phi_{ss} \tag{10}$$

where the local Mach number is defined as

$$M = \frac{\phi_s}{a} \gtreqless 1 \tag{11}$$

The FPE [Eq. (5)] in its final nonconservative canonical form[2] then becomes

$$\nabla \cdot (\rho V) = \rho[(1 - M^2)\phi_{ss} + \phi_{nn}] = 0 \tag{12}$$

The FPE is a homogeneous, quasilinear, partial differential equation of mixed elliptic-hyperbolic type and represents an exact nondissipative analytical model that conserves mass and energy but does not satisfy momentum conservation. Instead, it implicitly satisfies the constant entropy condition throughout the flowfield.

Artificial Density or Viscosity Concept (ADV)

The artificial density[4] or artificial viscosity[2,3] concept of generating the artificial dissipation in a locally supersonic region is generally formulated as

$$\tilde{\rho} = \rho - C\tilde{\mu}\rho_s \tag{13}$$

Here, C = const having the units of length, and $\tilde{\mu}$ is an appropriate switching function. The derivative of density must be performed in the locally upstream direction. Modified mass conservation then becomes

$$\nabla \cdot (\tilde{\rho}\nabla\phi) = \left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial n}\hat{e}_n\right)\cdot[(\rho - C\tilde{\mu}\rho_s)\phi_s\hat{e}_s + (\rho - C\tilde{\mu}\rho_s)\phi_n\hat{e}_n] \tag{14}$$

Hence

$$\nabla \cdot (\tilde{\rho}\nabla\phi) = \rho_s\phi_s + \rho\phi_{ss} - C\tilde{\mu}_s\rho_s\phi_s - C\tilde{\mu}\rho_{ss}\phi_s - C\tilde{\mu}\rho_s\phi_{ss} + \rho_n\phi_n + \rho\phi_{nn} - C\tilde{\mu}_n\rho_s\phi_n - C\tilde{\mu}\rho_s\phi_{nn} - C\tilde{\mu}\rho_{sn}\phi_n \tag{15}$$

Since $\phi_n = 0$, it follows that

$$\nabla \cdot (\tilde{\rho}\nabla\phi) = \rho\left\{\left[\phi_{ss} + \frac{\rho_s}{\rho}\phi_s + \phi_{nn}\right] - C\left[\tilde{\mu}\left(\frac{\rho_{ss}}{\rho}\phi_s + \frac{\rho_s}{\rho}\phi_{ss} + \frac{\rho_s}{\rho}\phi_{nn}\right) + \tilde{\mu}_s\frac{\rho_s}{\rho}\phi_s\right]\right\} \tag{16}$$

Table 1 Summary of the most prominent forms of the artificial dissipation based on the modified density formulation

	Modified density formulation	Reference
1	$\tilde{\rho} = \rho - \left(1 - \frac{1}{M^2}\right)\rho_s$	Hafez et al.[4] (and Jameson and Caughey[2,3] see Appendix A)
2	$\tilde{\rho} = -C\left(1 - \frac{1}{M^2}\right)\rho_s$	Amara et al.[7] $1.8 \leqq C \leqq 2.2$
3	$\tilde{\rho} = \rho - \left(1 - \frac{M_c^2}{M^2}\right)\rho_s$	Jameson[15] (see Appendix A) $0.8 \leqq M_c^2 < 1.0$
4	$\tilde{\rho} = \rho - C(M^2 - 1)\rho_s$	Roach and Sankar;[6] Xu et al.[8] $C = 2.0$ $0.1 \leqq C \leqq 0.6$
5	$\tilde{\rho} = \frac{(1-\omega)}{(\theta-1)^2}(x-1)(2\theta - 1 - x) + \omega$	Amara et al.[7] $2.0 \leqq \theta \leqq 3.0$ $1.0 \leqq x \leqq \theta$ $0.0 \leqq \omega \leqq 1.0$
6	$\tilde{\rho} = \rho - C(1 - \rho^n)\rho_s$	Sherif and Hafez[16] $C < 1$, $n = 1$

From Eq. (10) it follows (since $\phi_n = 0$) that

$$\rho_{ss} = -\frac{1}{a^4}\left\{[\rho_s\phi_s\phi_{ss} + \rho(\phi_{ss})^2 + \rho\phi_s\phi_{sss}]a^2 - \rho\phi_s\phi_{ss}\left(-\frac{\gamma-1}{2}\right)2\phi_s\phi_{ss}\right\} \tag{17}$$

Using Eq. (10) and Eq. (11) in Eq. (17) results in

$$\frac{\rho_{ss}}{\rho}\phi_s = -M^2\phi_{sss} + M^2[(2-\gamma)M^2 - 1]\frac{(\phi_{ss})^2}{\phi_s} \tag{18}$$

The artificially dissipative FPE, i.e., mass conservation equation based on the ADV concept, then assumes its most general form

$$\nabla\cdot(\tilde{\rho}\nabla\phi) = \rho[(1-M^2)\phi_{ss} + \phi_{nn}] + E_{\text{ADV}} = 0 \tag{19}$$

A common perception is that E_{ADV} contains only the term ϕ_{sss}.[4] This term produces linear dissipation; hence the expression artificial viscosity. The actual content of the term E_{ADV} has never been correctly analytically determined.[9,10] From Eqs. (10), (16), and (18) it follows that the most general exact analytic form of E_{ADV} is

$$E_{\text{ADV}} = C\rho\left\{\tilde{\mu}M^2\phi_{sss} - \tilde{\mu}M^2[(2-\gamma)M^2 - 2] - \frac{(\phi_{ss})^2}{\phi_s} + \tilde{\mu}M^2\frac{\phi_{ss}\phi_{nn}}{\phi_s} + \tilde{\mu}_s M^2\phi_{ss}\right\} \tag{20}$$

for any arbitrary switching function $\tilde{\mu}$.

The conventional form of the switching function $\tilde{\mu}$ used in ADV concepts can be generalized as

$$\tilde{\mu} = \left(1 - \frac{M_c^2}{M^2}\right)(M^2)^n \tag{21}$$

where M_c is the cutoff[15] Mach number ($M_c^2 \leqq 1$) and n is an integer (Table 1). This expression for $\tilde{\mu}$ is deduced from the form of truncation errors resulting when applying locally upstream differentiation to the term ϕ_{ss} that multiplies the $(1 - M^2)$ term in the FPE [Eq. (12)]. For the generalized conventional value of $\tilde{\mu}$ given by Eq. (21), it is now possible to resolve analytically the corresponding artificial dissipation, actually the error term E_{ADV}, given by Eq. (20). Since

$$M^2 = \frac{(\phi_s)^2 + (\phi_n)^2}{a^2} \tag{22}$$

with the help of Eq. (6), it follows that

$$(M^2)_s = \frac{1}{a^4}\left[2\phi_s\phi_{ss}a^2 - (\phi_s)^2\left(-\frac{\gamma-1}{2}\right)2\phi_s\phi_{ss}\right] \tag{23}$$

This can also be written with the help of Eq. (6) as

$$(M^2)_s = \phi_{ss}\left\{2\frac{\phi_s}{a^4}\left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2}(\phi_s)^2\right] + (\gamma-1)\frac{(\phi_s)^3}{a^4}\right\} \tag{24}$$

Finally

$$(M^2)_s = (\gamma+1)\frac{\phi_s}{a^4}\phi_{ss} = \frac{M^2}{\phi_s}[2 + (\gamma-1)M^2]\phi_{ss} \tag{25}$$

so that the generalized conventional formulation [Eq. (21)] for $\tilde{\mu}$ gives

$$\tilde{\mu}_s = (\gamma+1)\frac{\phi_{ss}}{\phi_s}\frac{(M^2)^{n-1}}{a^2}[M_c^2 + n(M^2 - M_c^2)] \tag{26}$$

Thus, with the conventional formula for the switching function $\tilde{\mu}$ (Eq. (21)], the artificially dissipative FPE [Eqs. (19) and (20)] and based on the ADV concept assumes the following exact general analytic form:

$$\nabla\cdot(\tilde{\rho}\nabla\phi) = \rho\Bigg([(1-M^2)\phi_{ss} + \phi_{nn}] + C(M^2 - M_c^2)(M^2)^n\phi_{sss} + C(M^2 - M_c^2)(M^2)^n\frac{\phi_{ss}\phi_{nn}}{\phi_s} + C\left\{(M^2 - M_c^2)[2 - (2-\gamma)M^2] + \frac{(\gamma-1)}{a^2} \times [M_c^2 + n(M^2 - M_c^2)]\right\}(M^2)^n\frac{(\phi_{ss})^2}{\phi_s}\Bigg) \tag{27}$$

The full effect of using the conventional formulations for the generalized switching function $\tilde{\mu}$ [Eq. (24)] is now available for inspection. Actually, there were several additional attempts at creating a better model for the artificial dissipation. One such attempt[7] uses a model that involves local grid spacing behind the shock wave. The model used by Sherif and Hafez[16] uses a switching function (Table 1) of the type $\tilde{\mu} = 1 - \rho$. With the help of Eqs. (10) and (20) it can be seen that this results in the error term of the form

$$E_{\text{ADV}} = C\rho\bigg((1-\rho)M^2\phi_{sss} + (1-\rho)M^2\frac{\phi_{ss}\phi_{nn}}{\phi_s} + M^2\{[(2-\gamma)M^2-2](1-\rho)+\rho M^2\}\frac{(\phi_{ss})^2}{\phi_s}\bigg) \tag{28}$$

Consequently, the following questions could be asked: 1) what are the effects of the artificial terms on the solution of exact nondissipative FPE [Eq. (12)]; 2) do these terms have effects similar to the physical viscous dissipation; and 3) what should be the appropriate form of the switching function $\tilde{\mu}$ that will make the artificially dissipative FPE [Eq. (27)] look as much as possible as an exact physically dissipative potential transonic flow equation?

One candidate for a physical dissipative model is the small perturbation viscous-transonic (V-T) equation that was derived by Cole,[17] Sichel,[18] and Ryzhov and Shefter.[19] It can be expressed as

$$-(\gamma-1)\varphi_x\varphi_{xx} + \varphi_{yy} + \frac{1}{R_e}\left(1+\frac{\gamma-1}{P_r''}\right)\varphi_{xxx} = 0 \tag{29}$$

which is a combination of the small perturbation transonic potential equation and Burgers equation.[18] It includes certain aspects of the heat conductivity and the longitudinal viscosity of the gas. Here, P_r'' is the Prandtl number based on longitudinal viscosity μ'', φ the velocity perturbation potential, and R_e the Reynolds number. This equation was successfully numerically integrated by Chin[20] and Sator.[21]

Actually, a more complete, nonlinear, PDP flow equation was derived recently by Dulikravich and Kennon.[14] They combined mass, momentum, and energy conservation equations into a single mass conservation for a calorically perfect gas

$$\begin{aligned}\rho\left\{\frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{a^2}V\cdot\frac{\partial V}{\partial t} + \left[(\nabla\cdot V) - \frac{1}{a^2}(V\cdot\nabla)\frac{V\cdot V}{2}\right]\right\} &= -\frac{1}{a^2}\Big(\rho V\cdot[V\times(\nabla\times V)] + V\cdot\{2(\nabla\cdot\mu\nabla)V \\ &\quad + \nabla\times[\mu(\nabla\times V)] + \nabla[\lambda(\nabla\cdot V)]\}\Big) \\ &\quad + \frac{\gamma-1}{a^2}[\Phi + \nabla\cdot(k\nabla T)]\end{aligned} \tag{30}$$

Here, Φ is the viscous dissipation function

$$\Phi = 2\mu\left\{\nabla\cdot[(V\cdot\nabla)V] + \frac{1}{2}(\nabla\times V)^2 - V\cdot\nabla(\nabla\cdot V)\right\} + \lambda(\nabla\cdot V)^2 \tag{31}$$

From the expanded Crocco-Wazsonyi equation[14] for viscous compressible fluids

$$T\nabla s - \nabla h_o = -V\times(\nabla\times V) + \frac{\partial V}{\partial t} - \frac{1}{\rho}\{2\nabla(\mu\nabla\cdot V) - \nabla\times[\mu(\nabla\times V)] + \nabla[\lambda(\nabla\cdot V)]\} \tag{32}$$

where s is the specific entropy and h_o the specific stagnation enthalpy; they have derived the vector operator form of the PDP equation

$$\begin{aligned}\rho\left\{\nabla^2\phi - \frac{1}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial(\nabla\phi\cdot\nabla\phi)}{\partial t} + (\nabla\phi\cdot\nabla)\left(\frac{\nabla\phi\cdot\nabla\phi}{2}\right)\right]\right\} &= -\frac{1}{a^2}\nabla\phi\cdot[\nabla(\mu''\nabla^2\phi)] \\ &\quad + \frac{\gamma-1}{a^2}\Big(2\mu\{\nabla\cdot[(\nabla\phi\cdot\nabla)\nabla\phi] - (\nabla\phi\cdot\nabla)(\nabla\cdot\nabla\phi)\} \\ &\quad + \lambda(\nabla^2\phi)^2\Big) + \frac{\gamma-1}{2}\nabla\cdot(k\nabla T) - \frac{1}{2}\mu''\frac{\partial}{\partial t}\nabla^2\phi\end{aligned} \tag{33}$$

The nondimensional canonical form of this equation for two-dimensional, steady flows is

$$\begin{aligned}\rho[(1-M^2)\phi_{ss}+\phi_{nn}] + \frac{\mu''}{R_e}\bigg\{&\left(1+\frac{\gamma-1}{P_r''}\right)\frac{\phi_s}{a^2}(\phi_{sss}+\phi_{snn}) \\ &-(\gamma-1)\left(1-\frac{1}{P_r''}\right)\left(\frac{(\phi_{ss})^2+(\phi_{nn})^2}{a^2}\right) \\ &-(\gamma-1)2\left(1-2\frac{\mu}{\mu''}\right)\frac{\phi_{ss}\phi_{nn}}{a^2} \\ &+(\gamma-1)2\left(\frac{1}{P_r''}-2\frac{\mu}{\mu''}\right)\frac{(\phi_{sn})^2}{a^2}\bigg\} = 0\end{aligned} \tag{34}$$

Here, $\mu = \mu/\mu_\infty$ is the shear viscosity coefficient, $\lambda = \lambda/\mu_\infty$ the secondary viscosity coefficient, $\mu'' = 2\mu + \lambda$ the longitudinal[18] viscosity coefficient, $k = k/k_\infty$ the heat conductivity coefficient, C_p the coefficient of specific heat at constant pressure, and $P_r'' = C_p\mu''/k$ the Prandtl number based on longitudinal viscosity.[18] The coefficients μ, λ, and k are assumed to be constants. It is now obvious that the V-T equation [Eq. (29)] contains only the most dominant linear dissipation term, since all other nonlinear dissipation terms were omitted during the linearization process. Therefore, it would be appropriate to compare the corresponding terms in the artificially dissipative FPE [Eqs. (27) or (28)] and in the PDP [Eq. (34)] rather than in the linearized small perturbation V-T equation [Eq. (29)]. Consequently, the ratio of terms multiplying (ϕ_{sss}) in Eqs. (27) and (34) is (Fig. 3)

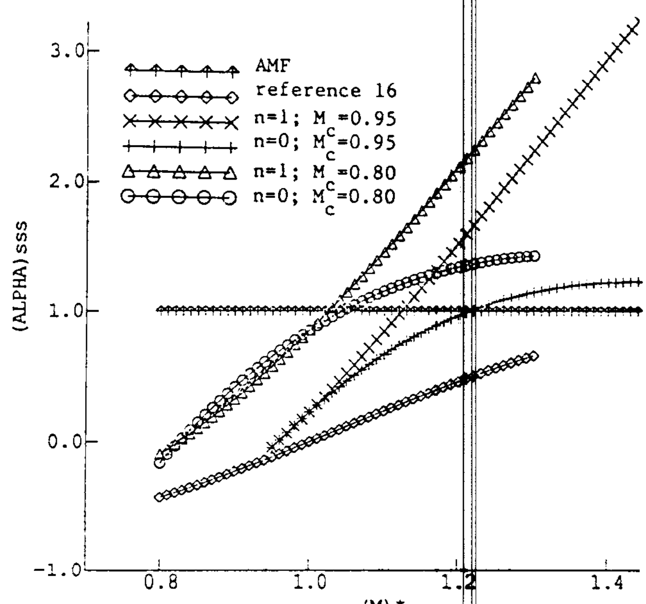

Fig. 3 Ratio of coefficients multiplying ϕ_{sss} terms in the one-dimensional versions of FPE with the corresponding coefficients in ADV and in the PDP equation: $(M_*)_1 = 1.2$, $\lambda/\mu = -2.118$; $\mu''/R_e = C = 0.00001985$; $P_r = 3/4$.

$$\alpha_{sss} = \frac{C\rho(M^2-M_c^2)(M^2)^n}{\frac{\mu''}{R_e}\left(1+\frac{\gamma-1}{P_r''}\right)\frac{\phi_s}{a^2}} = \frac{C}{\frac{\mu''}{R_e}\left(1+\frac{\gamma-1}{P_r''}\right)} \times \left[\frac{(M_c^2-M^2)}{\phi_s}(M^2)^n a^2\rho\right] \tag{35}$$

The ratio of terms multiplying $(\phi_{ss}\phi_{nn})$ in Eqs. (27) and (34) is (Fig. 4)

$$\alpha_{ssnn} = \frac{C\rho\,\frac{(M^2-M_c^2)(M^2)^n}{\phi_s}}{-2(\gamma-1)\frac{\mu''}{R_e}\frac{1}{a^2}\left(1-2\frac{\mu}{\mu''}\right)} = \left[\frac{-C}{2(\gamma-1)\frac{\mu''}{R_e}\left(1-2\frac{\mu}{\mu''}\right)}\right]\left[\frac{(M^2-M_c^2)}{\phi_s}(M^2)^n a^2\rho\right] \tag{36}$$

The ratio of terms multiplying $(\phi_{ss})^2$ in Eqs. (27) and (34) is (Fig. 5)

$$a_{ss}^2 = \frac{C\left\{[2-(2-\gamma)M^2] + \frac{(\gamma+1)}{a^2}\frac{M_c^2+n(M^2-M_c^2)}{(M^2-M_c^2)}\right\}}{-\frac{\mu''}{R_e}\left(1-\frac{1}{P_r''}\right)(\gamma-1)} \times \left[\frac{(M^2-M_c^2)}{\phi_s}(M^2)^n a^2\rho\right] \tag{37}$$

It would be ideal to have $\alpha_{sss}=1$, $\alpha_{ssnn}=1$, and $a_{ss}^2=1$ over the entire range of Mach numbers. Nevertheless, from this comparison it is clear that the ADV concepts[2] both generate terms that do not have the same magnitude and often not even the same sign as the physical dissipation terms. The true nature and effect of the introduction of the cutoff Mach number M_c can also be analyzed.

A somewhat different formulation is known as the DFB scheme,[11-13] which modifies the density according to

$$\tilde{\rho}_{\rm DFB} = \rho - \frac{1}{\phi_s}\{\rho[(\phi_s)^2+(\phi_n)^2]^{\frac{1}{2}}\}_s \tag{38}$$

Actually, it can be shown that the DFB formulation is equivalent to the ADV formulation (Appendix B).

Artificial Mass Flux Concept (AMF)

Instead of using the artificial density (compressibility)[4] or the artificial viscosity[2] formulations, the AMF[9,10] concept is hereby suggested as an alternative. The basic idea is to upstream differentiate not only the density but the entire streamwise mass flux at every supersonic point. The objective of the AMF formulation is to duplicate the physical dissipation as closely as possible.

The general AMF formulation applied to the mass conservation can be written as

$$\nabla\cdot(\overline{\rho V}) = \left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial n}\hat{e}_n\right)\cdot\{[(\rho\phi_s) - C\nu(\rho\phi_s)_s]\hat{e}_s + [\rho\phi_n]\hat{e}_n\} = 0 \tag{39}$$

From Eqs. (12) and (39) it follows that

$$\nabla\cdot(\overline{\rho V}) = \left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial n}\hat{e}_n\right)\cdot\left\{\left[(\rho\phi_s) - C\nu\left(\rho_s\phi_s - \frac{\phi_s}{M^2}\rho_s\right)\right]\hat{e}_s + [\rho\phi_n]\hat{e}_n\right\} = 0 \tag{40}$$

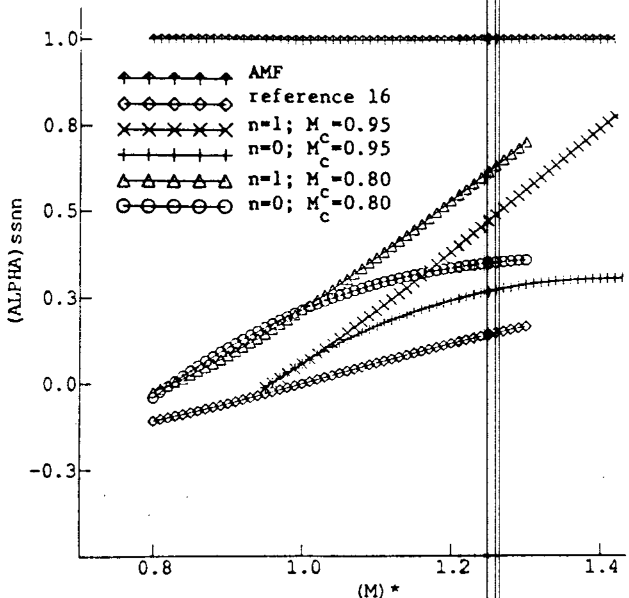

Fig. 4 Ratio of coefficients multiplying $\phi_{ss}\phi_{nn}$ terms in the one-dimensional versions of FPE with the corresponding coefficients in ADV and in the PDP equation: $(M^*)_t = 1.2$, $\lambda/\mu = -2.118$; $\mu''/R_e = C = 0.00001985$; $P_r = 3/4$.

or, finally,

$$\nabla\cdot(\overline{\rho V}) = (\tilde{\rho}\phi_s)_s + (\rho\phi_n)_n + E_{\rm AMF} = \rho[(1-M^2)\phi_{ss} + \phi_{nn}] \tag{41}$$

where

$$\tilde{\rho} = \rho - C\nu\left(1-\frac{1}{M^2}\right)\rho_s = \rho - C\tilde{\mu}\rho_s \tag{42}$$

and the AMF switching function $\tilde{\mu}$ is defined as

$$\tilde{\mu} = \nu\left(1-\frac{1}{M^2}\right) \tag{43}$$

The exact general analytic form of the error ("artificial dissipation") term $E_{\rm AMF}$ [Eqs. (20) and (41)] then becomes

$$E_{\rm AMF} = C\rho\left\{\tilde{\mu}M^2\phi_{sss} - \tilde{\mu}M^2[(2-\gamma)M^2-2]\frac{(\phi_{ss})^2}{\phi_s} + \tilde{\mu}M^2\frac{\phi_{ss}\phi_{nn}}{\phi_s} + \tilde{\mu}_s M^2\phi_{ss}\right\} \tag{44}$$

From Eq. (43) it follows that

$$\tilde{\mu}_s = \nu_s\left(1-\frac{1}{M^2}\right) + \nu\frac{(M^2)_s}{M^4} \tag{45}$$

With the help of Eq. (25) this becomes

$$\tilde{\mu}_s = \nu_s\left(1-\frac{1}{M^2}\right) + \nu(\gamma+1)\frac{\phi_{ss}}{\phi_s^3} \tag{46}$$

As a result, the AMF concept produces the following general form of the modified FPE:

$$\nabla\cdot(\overline{\rho V}) = \rho[(1-M^2)\phi_{ss}+\phi_{nn}] + C\rho\left(\nu(M^2-1)\phi_{sss} - \nu\left\{(M^2-1)[(2-\gamma)M^2-2] + \frac{(\gamma+1)}{\phi_s}\right\}\frac{(\phi_{ss})^2}{\phi_s} + \nu(M^2-1)\frac{\phi_{ss}\phi_{nn}}{\phi_s} + \nu_s(M^2-1)\phi_{ss}\right) \tag{47}$$

The formulation of ν could be deduced in a number of ways.[9,10] Since the main objective of the AMF is to make the coefficients multiplying the ϕ_{sss} term have the same sign and magnitude in both Eq. (47) (artificial dissipation) and in Eq. (34) (physical dissipation), then the adequate value for ν should be

$$\nu = \frac{\dfrac{\mu''}{R_e}\left(1 + \dfrac{\gamma - 1}{P_r''}\right)\dfrac{\phi_s}{a^2}}{C\rho(M^2 - 1)} = \frac{A\phi_s}{C\rho a^2(M^2 - 1)} \tag{48}$$

where

$$A = \frac{\mu''}{R_e}\left(1 + \frac{\gamma - 1}{P_r''}\right) \tag{49}$$

The AMF switching function consequently becomes

$$\tilde{\tilde{\mu}} = \nu\left(1 - \frac{1}{M^2}\right) = \frac{A\phi_s}{C\rho a^2 M^2} = \frac{A}{C}\frac{1}{\rho\phi_s} \tag{50}$$

Notice also that the AMF concept [Eqs. (41), (42), (49), and (12)] creates a familiar form of artificial density:

$$\tilde{\tilde{\rho}} = \rho - C\tilde{\tilde{\mu}}\rho_s = \rho - A\frac{\rho_s}{\phi_s\rho} = \rho + A\frac{\phi_{ss}}{a^2} \tag{51}$$

The nonphysical terms arising from the AMF formulation can now be written as

$$E_{\mathrm{AMF}} = A\left\{\frac{\phi_s}{a^2}\phi_{sss} + \frac{\phi_{ss}\phi_{nn}}{a^2} + \left[\frac{(\gamma + 1)}{a^2(M^2 - 1)} - (2 - \gamma)M^2 - 2\right]\frac{(\phi_{ss})^2}{a^2} + \rho\nu_s(M^2 - 1)\frac{C}{A}\phi_{ss}\right\} \tag{52}$$

From Eq. (48) it follows that

$$\nu_s = \frac{A}{C}\left\{\frac{\phi_{ss}\rho a^2(M^2 - 1) - [\phi_s\rho_s a^2(M^2 - 1) + \phi_s\rho(a^2)_s]}{[\rho a^2(M^2 - 1)]^2} + \frac{(M^2 - 1) + \phi_s\rho a^2(M^2)_s}{[\rho a^2(M^2 - 1)]^2}\right\} \tag{53}$$

Implementation of Eqs. (10) and (25) [together with the fact that $(\phi_s)^2 = a^2M^2$] in Eq. (53) results in

$$\nu_s = \frac{A}{C}\frac{\phi_{ss}}{\rho a^2(M^2 - 1)}\left[1 + \gamma M^2 - \frac{(\gamma + 1)}{(M^2 - 1)}\frac{M^2}{a^2}\right] \tag{54}$$

Introducing Eq. (54) in Eq. (52) results in the desired form of the AMF formulation

$$\nabla\cdot\overline{(\rho V)} = \rho[(1 - M^2)\phi_{ss} + \phi_{nn}] + A\left\{\frac{\phi_s}{a^2}\phi_{sss} + \frac{1}{a^2}\phi_{ss}\phi_{nn} + \left[3 + 2(\gamma - 1)M^2 - \frac{(\gamma + 1)}{a^2}\right]\frac{(\phi_{ss})^2}{a^2}\right\} \tag{55}$$

We can now perform the comparison of coefficients of the derivatives generated by the AMF concept [Eq. (55)] with the coefficients of the like derivatives in the PDP [Eq. (34)] equation:

$$\alpha_{sss} = \frac{A\dfrac{\phi_s}{a^2}}{\dfrac{\mu''}{R_e}\left(1 + \dfrac{\gamma - 1}{P_r''}\right)\dfrac{\phi_s}{a^2}} = 1 \tag{56}$$

$$a_{ssnn} = \frac{\dfrac{A}{a^2}}{-\dfrac{\mu''}{R_e}\dfrac{(\gamma - 1)}{a^2}2\left(1 - 2\dfrac{\mu}{\mu''}\right)} = 1 \tag{57}$$

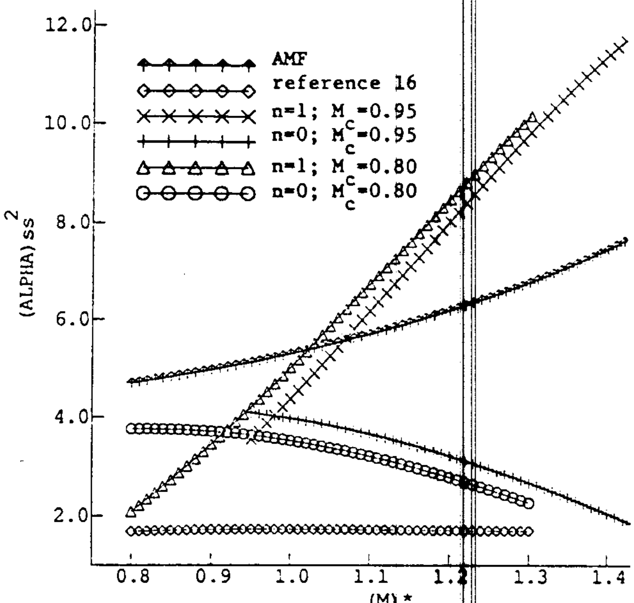

Fig. 5 Ratio of coefficients multiplying $(\phi_{ss})^2$ terms in the one-dimensional versions of FPE with the corresponding coefficients in ADV and in the PDP equation: $(M^*)_1 = 1.2$, $\lambda/\mu = -2.118$; $\mu''/R_e = C = 0.00001985$; $P_r = 3/4$.

From Eq. (57) it follows that

$$A = -\frac{\mu''}{R_e}(\gamma - 1)2\left(1 - 2\frac{\mu}{\mu''}\right) \tag{58}$$

Notice that the Prandtl number P_r'', based on longitudinal viscosity μ'', can be related to the Prandtl number P_r, based on the shear viscosity μ as follows:

$$\frac{1}{P_r''} = \frac{1}{P_r}\frac{\mu}{\mu''} \tag{59}$$

From Eqs. (49), (58), and (59), it follows that the condition for $\alpha_{ssnn} = 1$ is satisfied if

$$\mu'' = \frac{(\gamma - 1)\left(4 - \dfrac{1}{P_r}\right)}{1 + 2(\gamma - 1)} = 2 + \left[-2 - \frac{\gamma - 1}{P_r}\right]\Big/(2\gamma - 1) \tag{60}$$

Since the exact expression should be $\mu'' = 2\mu + \lambda$, Eq. (60) indicates that $\lambda = -[2 + (\gamma - 1)/P_r]/(2\gamma - 1)$ in order to make $\alpha_{ssnn} = 1$.

The problem arises, though, with the ratio of coefficients multiplying the $(\phi_{ss})^2$ term in Eqs. (55) and (34). Using Eq. (59), it follows that

$$\alpha_{ss}^2 = \frac{A\left[3 + 2(\gamma - 1)M^2 - \dfrac{\gamma + 1}{a^2}\right]\dfrac{1}{a^2}}{-\dfrac{\mu''}{R_e}\left(1 - \dfrac{1}{P_r}\dfrac{\mu}{\mu''}\right)\dfrac{(\gamma - 1)}{a^2}} \tag{61}$$

After introducing Eqs. (49), (60), and Eq. (6) in Eq. (61), it follows that the ratio of terms multiplying $(\phi_{ss})^2$ resulting from the AMF concept [Eq. (55)] and the terms multiplying (ϕ_{ss}^2) in the PDP flow equation [Eq. (34)] is

$$\alpha_{ss}^2 = \frac{4P_r + 2(\gamma - 1)}{1 + (\gamma - 1)(3 - 4P_r)}\,\frac{(\gamma + 1) + (\gamma - 1)(\phi_s)^2}{(\gamma + 1) - (\gamma - 1)(\phi_s)^2} \tag{62}$$

For diatomic gases ($\gamma = 7/5$ and $P_r = 3/4$) it follows that

$$\alpha_{ss}^2 = \frac{19}{5}\,\frac{6 + (\phi_s)^2}{6 - (\phi_s)^2} \tag{63}$$

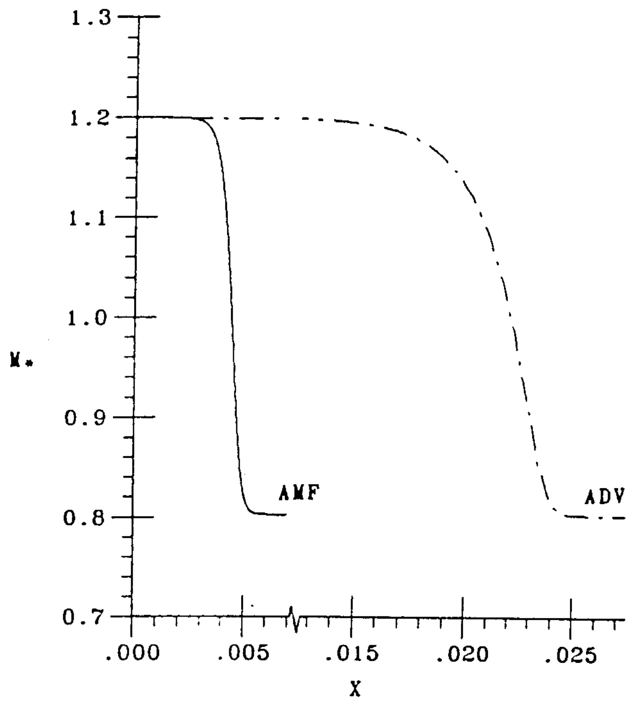

Fig. 6 Runge-Kutta solution of the one-dimensional form of the FPE with the ADV formulation ($C = 0.00001985$; $M_c = 0.77$, $\gamma = 7/5$) and with the AMF formulation ($\mu''/R_e = 0.00001985$; $P_r = 3/4$).

For monoatomic gases ($\gamma = 5/3$ and $P_r = 2/3$) it follows that

$$\alpha_{ss}^2 = \frac{36}{11}\frac{4+(\phi_s)^2}{4-(\phi_s)^2} \tag{64}$$

Thus, for AMF formulation the ratio of coefficients multiplying the $(\phi_{ss})^2$ term varies over the Mach number range (Fig. 5). Nevertheless, it has the correct sign. In addition, one-dimensional versions of the FPE with the ADV formulation and with the AMF formulation were integrated using the fourth-order Runge-Kutta scheme. The shock profiles indicate the shock symmetry resulting from the AMF formulation (Fig. 6).

Conclusions

Using strictly analytic tools, it was determined that the commonly used artificial density and artificial viscosity dissipation models for the numerical solution of the nondissipative FPE governing transonic steady flows produce a variety of additional terms. Some of these terms are of the same general type as the terms that exist in a PDP equation. Nevertheless, their coefficients have often entirely disproportionate magnitudes and signs, suggesting that the existing artificial dissipation models give seemingly accurate results because certain artificial dissipation terms compensate for some of the remaining artificial dissipation terms.

On the other hand, the AMF dissipation concept offers an alternative approach, since several of its terms can be matched with the corresponding terms in the PDP flow equation exactly. Moreover, the AMF formulation can be easily incorporated in the existing FPE solvers by introducing a new form of the switching function given by Eq. (51). The details are given in the Appendix C.

Appendix A

The artificial density[4] and the artificial viscosity[3] concepts are essentially the same[4] as confirmed by the following derivation.

Mass conservation with the artificial density can be expressed as

$$(\tilde{\rho}u)_x + (\tilde{\rho}v)_y = (\rho u + \tilde{Q})_x + (\rho v + \tilde{R})_y = 0 \tag{A1}$$

where

$$\tilde{Q} = -C\tilde{\mu}\rho_s u \tag{A2}$$

$$\tilde{R} = -C\tilde{\mu}\rho_s v \tag{A3}$$

From Eq. (10) it follows that

$$\rho_s = -\rho\frac{M^2}{\phi_s}\phi_{ss} = -\rho\frac{q}{a^2}\phi_{ss} \tag{A4}$$

It is also easy to show that

$$\phi_{ss} = (1/q^2)(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}) \tag{A5}$$

where

$$q^2 = u^2 + v^2 = \nabla\phi \cdot \nabla\phi \tag{A6}$$

Hence, the elements of the artificial density are

$$\tilde{Q} = (C\rho\tilde{\mu}/a^2)(u/q)(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}) \tag{A7}$$

$$\tilde{R} = (C\rho\tilde{\mu}/a^2)(v/q)(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}) \tag{A8}$$

The artificial viscosity[2,3] formulation uses the following truncated version:

$$\tilde{Q} = (C\rho\tilde{\mu}/a^2)(u^2\phi_{xx} + uv\phi_{xy}) \tag{A9}$$

$$\tilde{R} = (C\rho\tilde{\mu}/a^2)(uv\phi_{xy} + v^2\phi_{yy}) \tag{A10}$$

Appendix B

The directional flux biasing (DFB) scheme uses the following form of artificial density in the locally supersonic flow:

$$\tilde{\rho}_{\mathrm{DFB}} = (1/\phi_s)(\rho\phi_s) - \{\rho[(\phi_s)^2 + (\phi_n)^2]^{1/2}\}_s \tag{B1}$$

$$\tilde{\rho}_{\mathrm{DFB}} = \rho - (1/\phi_s)\{\rho_s\phi_s + \rho\tfrac{1}{2}[(\phi_s)^2 + (\phi_n)^2]^{-1/2}(2\phi_s\phi_{ss} + 2\phi_n\phi_{ns})\} \tag{B2}$$

Nevertheless, $\phi_n = 0$ by definition. Hence

$$\tilde{\rho}_{\mathrm{DFB}} = \rho - [\rho_s + \rho(\phi_{ss}/\phi_s)] = \rho - \left[\rho_s - \rho\left(\frac{\rho_s}{\rho}\frac{1}{M^2}\right)\right] \tag{B3}$$

Finally,

$$\tilde{\rho}_{\mathrm{DFB}} = \rho - [1 - (1/M^2)]\rho_s = \tilde{\rho}_{\mathrm{ADV}} \tag{B4}$$

Appendix C

The artificial mass flux (AMF) formulation [Eq. (41)] can be recast in the familiar artificial density form, i.e.,

$$(\tilde{\rho}\phi_s)_s + (\rho\phi_n)_n = 0 \tag{C1}$$

can be written as

$$\lfloor \nabla_{sn} \rfloor \begin{Bmatrix} \tilde{\rho} & \phi_s \\ \rho & \phi_n \end{Bmatrix} = \lfloor \nabla_{xy} \rfloor \begin{bmatrix} \frac{u}{q} & -\frac{v}{q} \\ \frac{v}{q} & \frac{u}{q} \end{bmatrix} \begin{Bmatrix} \tilde{\rho} & \phi_s \\ \rho & \phi_n \end{Bmatrix} \tag{C2}$$

or

$$\lfloor \nabla_{sn} \rfloor \begin{Bmatrix} \tilde{\tilde{\rho}} & \phi_s \\ \rho & \phi_n \end{Bmatrix} = \lfloor \nabla_{xy} \rfloor \begin{bmatrix} \frac{u}{q}\tilde{\tilde{\rho}}\phi_s & -\frac{v}{q}\rho\phi_n \\ \frac{v}{q}\tilde{\tilde{\rho}}\phi_s & \frac{u}{q}\rho\phi_n \end{bmatrix} \quad \text{(C3)}$$

Nevertheless, $\phi_n = 0$ and $\phi_s = q$. Hence

$$(\tilde{\tilde{\rho}}\phi_s)_s + (\rho\phi_n)_n = (\tilde{\tilde{\rho}}u)_x + (\tilde{\tilde{\rho}}v)_y - \left[\left(\frac{\rho v}{q}\phi_n\right)_x - \left(\frac{\rho u}{q}\phi_n\right)_y\right] \quad \text{(C4)}$$

Since

$$\phi_s = (1/q)(u\phi_x + v\phi_y) \quad \text{(C5)}$$

it follows that

$$\phi_{sx} = (1/q)(u\phi_{xx} + v\phi_{xy}) \quad \text{(C6)}$$

Similarly, since

$$\phi_n = (1/q)(-v\phi_x + u\phi_y) = 0 \quad \text{(C7)}$$

and Eq. (C6), it follows that

$$\phi_{nx} = \phi_{ny} = 0 \quad \text{(C8)}$$

Hence, the AMF can be expressed in a typical ADV form:

$$(\tilde{\tilde{\rho}}u)_x + (\tilde{\tilde{\rho}}v)_y = 0 \quad \text{(C9)}$$

where, after combining Eqs. (51), (58), and (60), it follows that

$$\tilde{\tilde{\rho}} = \rho - \frac{\mu''}{R_e}\frac{2(\gamma-1)}{a^2}\left[1 - 2\frac{1+2(\gamma-1)}{(\gamma-1)[4-(1/P_r)]}\right]\phi_{ss} \quad \text{(C10)}$$

or

$$\tilde{\tilde{\rho}} = \rho + \frac{1}{R_e}\frac{[2+(\gamma-1/P_r)]2(\gamma-1)}{1+2(\gamma-1)}\frac{\phi_{ss}}{a^2} \quad \text{(C11)}$$

or

$$\tilde{\tilde{\rho}} = \rho - \left[\frac{1}{R_e}\frac{[2+(\gamma-1/P_r)]2(\gamma-1)}{1+2(\gamma-1)}\frac{1}{\rho\phi_s}\right]\rho_s \quad \text{(C12)}$$

Thus, the AMF formulation requires only one physical input parameter besides the Prandtl number. This input parameter is the Reynolds number R_e.

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