

THEORY OF COMPRESSIBLE IRROTATIONAL FLOWS INCLUDING HEAT CONDUCTIVITY AND LONGITUDINAL VISCOSITY

G. S. DULIKRAVICH¹ and S. R. KENNON²

¹Aerospace Engineering Department, The Pennsylvania State University, University Park, PA 16802, U.S.A.

²Aerospace Engineering Department, University of Texas, Arlington, TX 76019, U.S.A.

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Abstract—A new exact analytical model was derived for the irrotational flows of compressible fluids when the effects of heat conductivity and molecular viscosity are allowed. This new model satisfies conservation of mass, momentum and energy exactly. In addition, it satisfies physical irrotationality conditions. Compared to the classical small perturbation viscous–transonic (V-T) equation, the new physically dissipative potential (PDP) equation contains a number of additional terms that are highly nonlinear. The new model is derived in a general vector operator form and in a scalar canonical form. The PDP equation is able to produce shock waves of different strengths depending on the ratio of secondary and shear viscosity coefficients. The one-dimensional, steady flow version of the PDP equation was integrated using a Runge–Kutta scheme and different values of the ratio of the two viscosities. The computed shock structures are symmetric. Rankine–Hugoniot shock jumps were obtained when Stokes’ hypothesis was used in the PDP equation and isentropic shock jumps were obtained when the longitudinal viscosity was negligible.

INTRODUCTION

An exact analytical model for nondissipative, irrotational, inviscid, heat nonconducting, compressible fluid flow is the full potential equation (FPE). During the late forties, Cole [1] derived a new analytic model for potential, steady, two-dimensional flows by partially incorporating heat conductivity and secondary viscosity effects. Linearization based on small perturbation theory is a possible reason why this original viscous–transonic (V-T) equation retains only the most essential physical nonlinearities. Actually, as clearly stated in the works of Sichel [2, 3], the V-T equation represents a combination of the classical transonic small perturbation potential flow equation which contains the most essential nonlinearities of inviscid flows, and the Burgers equation which contains the most essential linear dissipation effects. Ryzhov and Shefter [4] used physical arguments to justify small perturbation linearization processes used in the derivation of V-T equations for planar and axisymmetric flows. These authors succeeded also in obtaining analytic solutions for V-T equations governing transonic flows about thin airfoils, thin projectiles and through shock waves. Chin [5] successfully integrated the V-T equation numerically for a steady two-dimensional transonic flow around an isolated airfoil.

The objective of this work is to derive an exact physically dissipative potential (PDP) flow equation without resorting to linearizations. Thus, the intention is to create a physical model that is more complete than the classical V-T equation and the FPE, and which is based on a single dependent variable.

CONSERVATION LAWS

The equation of state for a thermally perfect gas

$$p = \rho RT \quad (1)$$

links thermodynamic static pressure, p , density, ρ , and absolute temperature, T . From equation (1) it follows that

$$\ln p = \ln \rho + \ln R + \ln T. \quad (2)$$

For a perfect gas, the specific gas constant is $R = \text{const.}$ Then, the total time derivative of the above equation becomes

$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{\rho} \frac{D\rho}{Dt} + \frac{1}{T} \frac{DT}{Dt}. \quad (3)$$

Mass conservation can be expressed as

$$\rho(\nabla \cdot \mathbf{V}) = -\frac{D\rho}{Dt} + \dot{m}, \quad (4)$$

where \dot{m} designates the rate of generation of mass per unit time per unit volume and \mathbf{V} is the local fluid velocity vector. After introduction of equations (3) and (1), the mass conservation equation becomes

$$\rho(\nabla \cdot \mathbf{V}) = \frac{\rho}{T} \frac{DT}{Dt} - \frac{1}{RT} \frac{Dp}{Dt} + \dot{m}. \quad (5)$$

Energy conservation can be expressed as

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \Phi + \nabla \cdot (k\nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} - \dot{m} \left(e + \frac{p}{\rho} - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right). \quad (6)$$

Here, e is the internal energy per unit mass, h is the static enthalpy per unit mass, k is the heat conduction coefficient of the fluid assuming Fourier's law, Φ is the viscous dissipation function, $\dot{\mathbf{q}}_r$ is the time rate of radiation heat flux vector and \dot{Q} is the time rate of internal heat generation.

The viscous dissipation function Φ is defined in its vector operator form as

$$\Phi = 2\mu \{ \nabla \cdot [(\nabla \cdot \nabla)\mathbf{V}] + \frac{1}{2}(\nabla \times \mathbf{V})^2 - (\nabla \cdot \nabla)(\nabla \cdot \mathbf{V}) \} + \lambda(\nabla \cdot \mathbf{V})^2, \quad (7)$$

where μ is the shear viscosity coefficient and λ is the secondary viscosity coefficient. For calorically perfect gases

$$h = C_p T = e + \frac{p}{\rho}, \quad (8)$$

where the specific heat at constant pressure, C_p , is constant. Therefore, equation (6) divided by $C_p T$ can be rewritten as

$$\frac{\rho}{T} \frac{DT}{Dt} = \frac{1}{C_p T} \frac{Dp}{Dt} + \frac{1}{C_p T} \left[\Phi + \nabla \cdot (k\nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} - \dot{m} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right]. \quad (9)$$

Substitution of equation (9) (energy conservation) in equation (5) (mass conservation) results in

$$\rho(\nabla \cdot \mathbf{V}) = \left(\frac{1}{C_p T} - \frac{1}{RT} \right) \frac{Dp}{Dt} + \frac{1}{C_p T} \left[\Phi + \nabla \cdot (k\nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} - \dot{m} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] + \dot{m}. \quad (10)$$

Note that

$$\left(\frac{1}{C_p T} - \frac{1}{RT} \right) = \left(\frac{\gamma - 1}{\gamma RT} - \frac{\gamma}{\gamma RT} \right) = -\frac{1}{a^2}, \quad (11)$$

where $\gamma = C_p/C_v$ and a is the local isentropic speed of sound. Then, equation (10) becomes

$$\rho(\nabla \cdot \mathbf{V}) = -\frac{1}{a^2} \frac{Dp}{Dt} + \frac{\gamma - 1}{a^2} \left[\Phi + \nabla \cdot (k\nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} - \dot{m} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] + \dot{m}. \quad (12)$$

Momentum conservation can be expressed as

$$\nabla p = \rho \mathbf{b} - \rho \frac{D\mathbf{V}}{Dt} - \dot{m} \mathbf{V} + \{ 2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu \nabla \times \mathbf{V}) + \nabla(\lambda \nabla \cdot \mathbf{V}) \}, \quad (13)$$

where \mathbf{b} is the body force per unit mass. Pre-multiplying equation (13) with \mathbf{V} and using the vector identity

$$(\nabla \cdot \nabla)\mathbf{V} = \nabla \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) - \mathbf{V} \times (\nabla \times \mathbf{V}), \quad (14)$$

it follows that the total differential

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p \quad (15)$$

can be written as

$$\begin{aligned} \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \rho \left[\mathbf{V} \cdot \mathbf{b} - \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V} \cdot \nabla) \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) + \mathbf{V} \cdot (\mathbf{V} \times (\nabla \times \mathbf{V})) \right] \\ + \mathbf{V} \cdot \{ 2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu \nabla \times \mathbf{V}) + \nabla(\lambda \nabla \cdot \mathbf{V}) \} - \dot{m} \mathbf{V} \cdot \mathbf{V}. \end{aligned} \quad (16)$$

Hence, mass conservation [equation (12)] which already includes energy conservation, becomes after inclusion of momentum conservation [equation (16)]:

$$\begin{aligned} \rho \left\{ \frac{1}{\rho a^2} \frac{\partial p}{\partial t} - \frac{1}{a^2} \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + \left[(\nabla \cdot \mathbf{V}) - \frac{1}{a^2} (\mathbf{V} \cdot \nabla) \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] \right\} \\ = \frac{-1}{a^2} \left\{ \rho [\mathbf{V} \cdot \mathbf{b} + \mathbf{V} \cdot (\mathbf{V} \times (\nabla \times \mathbf{V}))] + \mathbf{V} \cdot [2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu \nabla \times \mathbf{V}) + \nabla(\lambda \nabla \cdot \mathbf{V})] \right. \\ \left. - (\gamma - 1) \left[\Phi + \nabla \cdot (k \nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} - \dot{m} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] \right\} + \dot{m} (1 + M^2), \end{aligned} \quad (17)$$

Here, the local Mach number is defined as $M = |\mathbf{V}|/a$. Since $a^2 = \gamma R T$, note that

$$\dot{m} \left[-\frac{\gamma - 1}{a^2} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) + (1 + M^2) \right] = \dot{m} \left(-\frac{\gamma - 1}{\gamma R T} C_p T + \frac{\gamma - 1}{2} M^2 + 1 + M^2 \right). \quad (18)$$

Also, since

$$C_p = \frac{\gamma R}{\gamma - 1} \quad (19)$$

it follows that equation (18) can be rewritten as

$$\frac{\gamma - 1}{a^2} \left[-\dot{m} \left(h - \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] + \dot{m} (1 + M^2) = \dot{m} M^2 \left(\frac{\gamma + 1}{2} \right). \quad (20)$$

Hence, mass conservation [equation (17)] can be written as

$$\begin{aligned} \rho \left\{ \frac{1}{\rho a^2} \frac{\partial p}{\partial t} - \frac{1}{a^2} \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + \left[\nabla \cdot \mathbf{V} - \frac{1}{a^2} (\mathbf{V} \cdot \nabla) \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right] \right\} \\ = \frac{-1}{a^2} \left\{ \rho [\mathbf{V} \cdot \mathbf{b} + \mathbf{V} \cdot (\mathbf{V} \times (\nabla \times \mathbf{V}))] + \mathbf{V} \cdot [2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu \nabla \times \mathbf{V}) + \nabla(\lambda \nabla \cdot \mathbf{V})] \right\} \\ + \frac{\gamma - 1}{a^2} \left[\Phi + \nabla \cdot (k \nabla T) - \nabla \cdot \dot{\mathbf{q}}_r + \dot{Q} \right] + \dot{m} M^2 \left(\frac{\gamma + 1}{2} \right). \end{aligned} \quad (21)$$

Notice that equation (21) is an exact formula for mass conservation that also implicitly satisfies the exact momentum and energy conservation equations for a calorically perfect gas.

IRROTATIONALITY CONDITION

Gibbs relation expressed in its vector operator form as

$$T \nabla s - \nabla h = -\frac{1}{\rho} \nabla p \quad (22)$$

can be expanded by adding

$$\nabla \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right)$$

to both sides, i.e.

$$T\nabla s - \nabla \left(h + \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) = - \left[\frac{1}{\rho} \nabla p + \nabla \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) \right]. \quad (23)$$

Introduction of equations (13) and (14) into equation (23) results in an equation similar to the Crocco-Wazsonyi [6] equation:

$$T\nabla s - \nabla h_0 = -\mathbf{V} \times (\nabla \times \mathbf{V}) + \frac{\partial \mathbf{V}}{\partial t} - \mathbf{b} - \frac{1}{\rho} \{ 2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu(\nabla \times \mathbf{V})) + \nabla(\lambda \nabla \cdot \mathbf{V}) - \dot{m} \dot{\mathbf{V}} \}. \quad (24)$$

This equation relates thermodynamic quantities and flow kinematics. It is valid for unsteady flow of a calorically perfect, compressible, heat conducting, viscous fluid under the influence of body forces and allows for mass sources and sinks. Here, $h_0 = h + (\mathbf{V} \cdot \mathbf{V}/2)$ is the stagnation enthalpy per unit mass. If body forces, mass generation and unsteadiness are neglected, if the flow is assumed to be irrotational ($\nabla \times \mathbf{V} = 0$), and if μ , λ and k are assumed constant, then equation (24) becomes

$$T\nabla s = -\frac{\mu''}{\rho} \nabla(\nabla \cdot \mathbf{V}) + \nabla h_0, \quad (25)$$

where $\mu'' = 2\mu + \lambda$ is the longitudinal [3] viscosity coefficient. This means that the flow can be potential ($\mathbf{V} = \nabla\phi$), although nonisentropic and that the entire flow field can be described with a single variable called the velocity potential function ϕ . This general concept of nonisentropic potential flows was clearly described by Klopfer and Nixon [7]. Actually, "the assumption of irrotational flow, which is a key step in the present development, cannot be rigorously justified *a priori*" [3]. Thus, the following derivation is "based on the concept of a fluid which has only compression viscosity so that it can still slip over the airfoil surface as in inviscid flow" [3].

THE PDP FLOW EQUATION

After neglecting body forces and mass generation, momentum conservation [equation (13)] becomes

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \{ 2\mu \nabla(\nabla \cdot \mathbf{V}) - \mu(\nabla \times (\nabla \times \mathbf{V})) + \lambda \nabla(\nabla \cdot \mathbf{V}) \}. \quad (26)$$

Using the vector identity [equation (14)] in equation (26) and assuming that the flow is irrotational results in

$$-\rho \nabla \left[\frac{\partial \phi}{\partial t} + \frac{\nabla \phi \cdot \nabla \phi}{2} \right] = \nabla [p - \mu'' \nabla^2 \phi]. \quad (27)$$

By multiplying both sides with a unit vector collinear with the gradient, equation (27) will become a scalar equation. After taking partial derivatives of both sides with respect to time and dividing both sides by a^2 , it follows that

$$-\frac{1}{a^2} \left[\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} \left(\frac{\nabla \phi \cdot \nabla \phi}{2} \right) \right] = \frac{1}{\rho a^2} \frac{\partial p}{\partial t} - \frac{1}{\rho a^2} \frac{\partial}{\partial t} (\mu'' \nabla^2 \phi). \quad (28)$$

Finally,

$$\frac{1}{\rho a^2} \frac{\partial p}{\partial t} - \frac{1}{a^2} \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{a^2} \left[\frac{\partial^2 \phi}{\partial t^2} + \nabla \phi \cdot \frac{\partial}{\partial t} (\nabla \phi) + \frac{\partial}{\partial t} \left(\frac{\nabla \phi \cdot \nabla \phi}{2} \right) \right] + \frac{1}{\rho a^2} \frac{\partial}{\partial t} (\mu'' \nabla^2 \phi) \quad (29)$$

should be substituted in equation (21) when the flow is irrotational.

Hence, the general vector operator form of mass conservation [equation (21)] in the case of an irrotational flow of a calorically perfect gas allowing for heat conduction, shear viscosity and secondary viscosity (but neglecting body forces, radiation heat transfer and mass and heat sources)

becomes

$$\begin{aligned} & \rho \left\{ \nabla^2 \phi - \frac{1}{a^2} \left[\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial(\nabla \phi \cdot \nabla \phi)}{\partial t} + (\nabla \phi \cdot \nabla) \left(\frac{\nabla \phi \cdot \nabla \phi}{2} \right) \right] \right\} \\ &= -\frac{1}{a^2} \nabla \phi \cdot \nabla (\mu'' \nabla^2 \phi) + \frac{\gamma - 1}{a^2} k \nabla^2 T - \frac{\mu''}{a^2} \frac{\partial}{\partial t} \nabla^2 \phi \\ &+ \frac{\gamma - 1}{a^2} \{ 2\mu [\nabla \cdot ((\nabla \phi \cdot \nabla) \nabla \phi) - (\nabla \phi \cdot \nabla)(\nabla \cdot \nabla \phi)] + \lambda (\nabla^2 \phi)^2 \}. \end{aligned} \tag{30}$$

CANONICAL FORM OF THE PDP EQUATION

The above equation can be expressed in a locally streamline aligned [8] cartesian coordinate system (*s, m, n*). Here, *s* is the streamline direction and *n* and *m* form a plane perpendicular locally to the streamline. The velocity components normal to the streamlines are zero ($\phi_m = \phi_n = 0$) by definition. By introducing the constant coefficient of longitudinal [9] viscosity $\mu'' = 2\mu + \lambda$, it follows that equation (30) transforms to

$$\begin{aligned} & \rho \left\{ (\phi_{ss} + \phi_{mm} + \phi_{nn}) - \frac{1}{a^2} (\phi_{tt} + 2\phi_s \phi_{st}) - \frac{1}{a^2} \left(\phi_s \frac{\partial}{\partial s} + \phi_m \frac{\partial}{\partial m} + \phi_n \frac{\partial}{\partial n} \right) \left(\frac{\phi_s^2 + \phi_m^2 + \phi_n^2}{2} \right) \right\} \\ &= -\frac{\phi_s}{a^2} \mu'' (\phi_{sss} + \phi_{mms} + \phi_{nns}) - \frac{\mu''}{a^2} (\phi_{sst} + \phi_{mmt} + \phi_{nnt}) \\ &+ \frac{\gamma - 1}{a^2} \mu'' (\phi_{ss}^2 + \phi_{mm}^2 + \phi_{nn}^2 + 2\phi_{ss} \phi_{mm} + 2\phi_{ss} \phi_{nn} + 2\phi_{mm} \phi_{nn}) \\ &+ \frac{\gamma - 1}{a^2} 4\mu (\phi_{sm}^2 + \phi_{sn}^2 + \phi_{mn}^2 - \phi_{ss} \phi_{mm} - \phi_{ss} \phi_{nn} - \phi_{mm} \phi_{nn}) + \frac{\gamma - 1}{a^2} k (T_{ss} + T_{mm} + T_{nn}). \end{aligned} \tag{31}$$

All quantities will be normalized with their critical thermodynamic values. The absolute temperature normalized with the critical temperature is

$$T = \frac{\bar{T}}{\bar{T}_*} = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} (\phi_s^2 + \phi_m^2 + \phi_n^2), \tag{32}$$

where the overbars designate dimensional quantities. Then

$$T_{ss} = -(\gamma - 1) (\phi_{ss}^2 + \phi_s \phi_{sss} + \phi_m^2 \phi_{ms} + \phi_m \phi_{mss} + \phi_n^2 \phi_{ns} + \phi_n \phi_{nss}), \tag{33}$$

$$T_{mm} = -(\gamma - 1) (\phi_{sm}^2 + \phi_s \phi_{smm} + \phi_{mm}^2 + \phi_m \phi_{mmm} + \phi_{nm}^2 + \phi_n \phi_{nmm}) \tag{34}$$

and

$$T_{nn} = -(\gamma - 1) (\phi_{sn}^2 + \phi_s \phi_{snn} + \phi_{nn}^2 + \phi_m \phi_{mnn} + \phi_{nn}^2 + \phi_n \phi_{nnn}). \tag{35}$$

Since $\phi_m = \phi_n = 0$, $\phi_s = \bar{\phi}_s / \bar{a}_* = M_*$, $a^2 = \bar{a}^2 / \bar{a}_*^2$, $\rho = \bar{\rho} / \bar{\rho}_*$, $\mu = \bar{\mu} / \bar{\mu}_\infty$, $\mu'' = \bar{\mu}'' / \bar{\mu}_\infty$ and $k = \bar{k} / \bar{k}_\infty$, then the nondimensional mass conservation [equation (31)] can be written as

$$\begin{aligned} & \rho \left\{ [(1 - M^2) \phi_{ss} + \phi_{mm} + \phi_{nn}] - \frac{1}{a^2} (\phi_{tt} + 2\phi_s \phi_{st}) \right\} \\ &= \frac{1}{\text{Re}} \left\{ -\frac{\phi_s}{a^2} \left[\mu'' + (\gamma - 1) \frac{k}{C_p} \right] (\phi_{sss} + \phi_{smm} + \phi_{snn}) \right. \\ &+ \frac{\gamma - 1}{a^2} \left(\mu'' - \frac{k}{C_p} \right) (\phi_{ss}^2 + \phi_{mm}^2 + \phi_{nn}^2) + 2 \frac{\gamma - 1}{a^2} (\mu'' - 2\mu) (\phi_{ss} \phi_{mm} + \phi_{ss} \phi_{nn} + \phi_{mm} \phi_{nn}) \\ &\left. - 2 \frac{\gamma - 1}{a^2} \left(\frac{k}{C_p} - 2\mu \right) (\phi_{sm}^2 + \phi_{sn}^2 + \phi_{mn}^2) - \frac{\mu''}{a^2} (\phi_{sst} + \phi_{mmt} + \phi_{nnt}) \right\}. \end{aligned} \tag{36}$$

Since

$$\frac{(\gamma - 1)}{\gamma R} = \frac{1}{C_p},$$

it is convenient to define a longitudinal [3] Prandtl number Pr'' as

$$\frac{1}{Pr''} = \frac{\bar{k}_\infty}{\bar{C}_p \bar{\mu}''}$$

and the Reynolds number Re as

$$Re = \bar{\rho}_* \bar{a}_* \bar{L} / \bar{\mu}_\infty.$$

Hence, the nondimensional formula for mass conservation in an irrotational flow of a heat conducting, calorically perfect, viscous gas without body forces, radiation heat transfer and mass sources or sinks is

$$\begin{aligned} & \rho \left\{ [(1 - M^2)\phi_{ss} + \phi_{mm} + \phi_{nn}] - \frac{1}{a^2} (\phi_{tt} + 2\phi_s \phi_{st}) \right\} \\ &= \frac{\mu''}{Re} \left\{ -\frac{\phi_s}{a^2} \left(1 + \frac{\gamma - 1}{Pr''} \right) (\phi_{sss} + \phi_{snn}) + \frac{\gamma - 1}{a^2} \left(1 - \frac{1}{Pr''} \right) (\phi_{ss}^2 + \phi_{mm}^2 + \phi_{nn}^2) \right. \\ & \quad + 2 \frac{\gamma - 1}{a^2} \left(1 - \frac{2}{\mu''} \right) (\phi_{ss} \phi_{mm} + \phi_{ss} \phi_{nn} + \phi_{mm} \phi_{nn}) \\ & \quad \left. - 2 \frac{\gamma - 1}{a^2} \left(\frac{1}{Pr''} - \frac{2}{\mu''} \right) (\phi_{sm}^2 + \phi_{sn}^2 + \phi_{mn}^2) - \frac{1}{a^2} (\phi_{sst} + \phi_{mmt} + \phi_{nnt}) \right\}. \end{aligned} \quad (37)$$

Notice that μ'' in this equation is actually $\bar{\mu}''/\bar{\mu}$. We will refer to equation (37) as the PDP flow equation. If the flow is steady and two-dimensional, equation (37) reduces to

$$\begin{aligned} \rho [(1 - M^2)\phi_{ss} + \phi_{nn}] &= \frac{\mu''}{Re} \left\{ -\frac{\phi_s}{a^2} \left(1 + \frac{\gamma - 1}{Pr''} \right) (\phi_{sss} + \phi_{snn}) + \frac{\gamma - 1}{a^2} \left(1 - \frac{1}{Pr''} \right) (\phi_{ss}^2 + \phi_{nn}^2) \right. \\ & \quad \left. + 2 \frac{\gamma - 1}{a^2} \left(1 - \frac{2}{\mu''} \right) \phi_{ss} \phi_{nn} - 2 \frac{\gamma - 1}{a^2} \left(\frac{1}{Pr''} - \frac{2}{\mu''} \right) \phi_{sn}^2 \right\}. \end{aligned} \quad (38)$$

Again, note that equations (37) and (38) satisfy energy, momentum and mass conservation and that they were derived without the assumptions of small perturbations and the consequent linearizations. When the longitudinal viscosity is negligible, the entire r.h.s. becomes zero and equation (38) converts to a nondissipative FPE. This highly nonlinear expression can now be compared with the classical V-T small perturbation equation [3]

$$(K_v K_\infty - \tilde{\phi}_x) \tilde{\phi}_{xx} + \tilde{\phi}_{yy} = \frac{-K_v}{Re} \left(1 + \frac{\gamma - 1}{Pr''} \right) \tilde{\phi}_{xxx}, \quad (39)$$

where

$$K_v = [\tau(\gamma + 1)M_\infty^2]^{-2/3} \quad K_\infty = (1 - M_\infty^2)$$

and τ is half the thickness ratio used in the small perturbation theory.

A more complete model is known as the pseudo-transonic [9, 10] V-T equation,

$$[1 - M_\infty^2 - (\gamma + 1)M_\infty^4 \tilde{\phi}_x] \tilde{\phi}_{xx} + \tilde{\phi}_{yy} = -\delta \tilde{\phi}_{xxx}. \quad (40)$$

Here, $\delta > 0$ is a small diffusion coefficient and $\tilde{\phi}$ is the perturbation velocity potential: $|\tilde{\phi}_x| \ll 1$. It is obvious that although the linear dissipation terms in both equations are practically the same, the PDP equation is considerably more complex than any of the V-T equations, since the PDP equation retains all the nonlinearities.

ADIABATIC SHOCK CONDITIONS

The conservation of energy [equation (6)], with the assumption that $\dot{q}_r = \dot{Q} = \dot{m} = 0$, can be written as

$$(\mathbf{V} \cdot \nabla)h = -\frac{\partial h}{\partial t} + \frac{1}{\rho} \left\{ \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p + \Phi + \nabla \cdot (k \nabla T) \right\}. \quad (41)$$

The conservation of momentum [equation (13)] can be expressed as

$$\nabla p = -\rho \frac{\partial \mathbf{V}}{\partial t} - \rho (\mathbf{V} \cdot \nabla) \mathbf{V} + \{2\nabla(\mu \nabla \cdot \mathbf{V}) - \nabla \times (\mu \nabla \times \mathbf{V}) + \nabla(\lambda \nabla \cdot \mathbf{V})\} \quad (42)$$

if the body forces are negligible. Substituting equation (42) into equation (41), using the vector identity [equation (14)] and keeping only the steady terms, yields

$$\rho \mathbf{V} \cdot \nabla h_0 = \mathbf{V} \cdot \{ \mu'' \nabla(\nabla \cdot \mathbf{V}) - \mu \nabla \times (\nabla \times \mathbf{V}) \} + \Phi + k \nabla^2 T + \rho \mathbf{V} \cdot (\mathbf{V} \times (\nabla \times \mathbf{V})). \quad (43)$$

For one-dimensional flow (which is always irrotational), the nondimensional version of equation (43) reduces to

$$\frac{dh_0}{dx} = \frac{\gamma - 1}{\rho} \left(1 - \frac{1}{Pr''} \right) \left(\frac{\phi_{xx}^2}{\phi_x} + \phi_{xxx} \right) \frac{\mu''}{Re}. \quad (44)$$

Equation (44) indicates that for steady, one-dimensional flows the stagnation enthalpy, h_0 , remains constant through a shock wave only when $Pr'' = 1$ is satisfied. Since $Pr'' = Pr \mu''/\mu$ and $Pr = 3/4$ for a diatomic gas, it follows that this is true only when Stokes' hypothesis $\mu''/\mu = 4/3$ is used. Nevertheless, Stokes' hypothesis is correct for monoatomic gases only.

NUMERICAL EXAMPLES

With equations (44) and (25) the entropy variation through a normal shock can be found. This equation was numerically integrated assuming Stokes' hypothesis (Fig. 1). The final entropy jump across the shock wave satisfies the Rankine-Hugoniot jump condition [11] for entropy. Nevertheless, the entropy exhibits a sharp spike in the middle of the shock (Fig. 1). From the entropy generation equation, it is easy to explain this phenomena. The viscous dissipation Φ is always positive. The heat flux ($k \nabla^2 T$) is positive only until the middle of the shock; downstream from the middle of the shock it becomes negative, thus lowering the entropy.

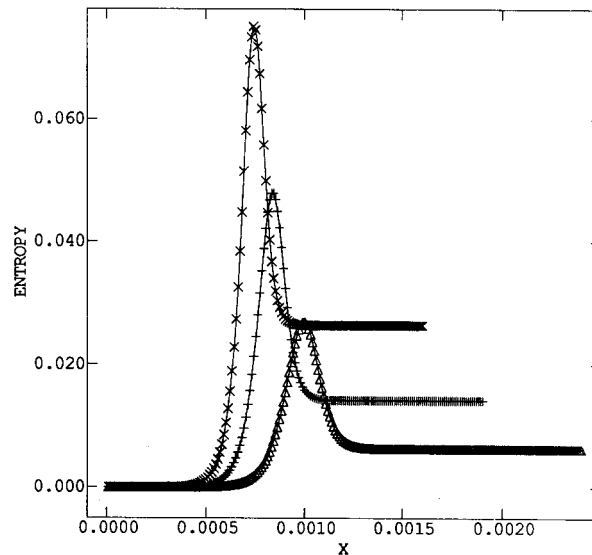


Fig. 1. Entropy variation through a one-dimensional shock produced by the PDP equation with $Pr = 3/4$, $\gamma = 1.4$, $\lambda/\mu = -2/3$, $Re = 10^5$ and: $(\phi_x)_1 = 1.15$ (\triangle — \triangle); $(\phi_x)_1 = 1.20$ ($+$ — $+$); $(\phi_x)_1 = 1.25$ (\times — \times).

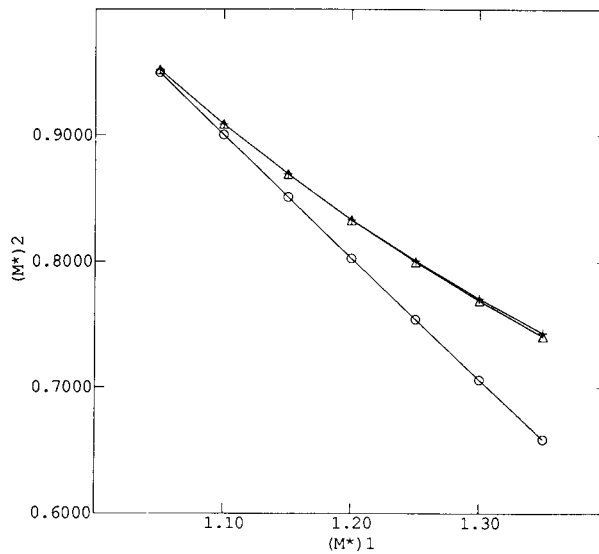


Fig. 2. One-dimensional shock jumps for the FPE (\circ — \circ), the PDP equation with $\lambda/\mu = -2/3$ (Δ — Δ) and for Rankine-Hugoniot shocks (+—+).

For the purpose of testing the accuracy and evaluating the sensitivity of the PDP equation (37), its one-dimensional steady version was used:

$$\rho(1 - M^2)\phi_{xx} = -\frac{\mu''}{\text{Re}} \left(1 + \frac{\gamma - 1}{\text{Pr}''}\right) \frac{\phi_s}{a^2} \phi_{xxx} + \frac{\mu''}{\text{Re}} \frac{\gamma - 1}{a^2} \left(1 - \frac{1}{\text{Pr}''}\right) \phi_{xx}^2. \quad (45)$$

Since

$$\rho(1 - M^2) = \frac{\rho}{a^2} (a^2 - \phi_x^2) \quad (46)$$

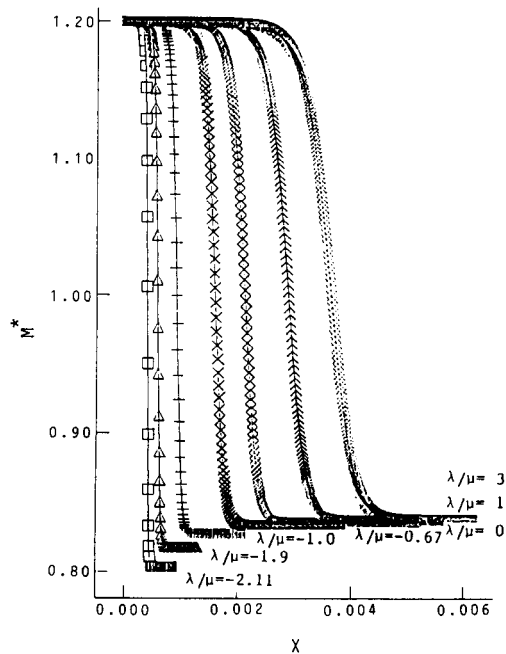


Fig. 3. Shock profiles computed with different values of λ/μ . Notice the varying shock strength and thickness.

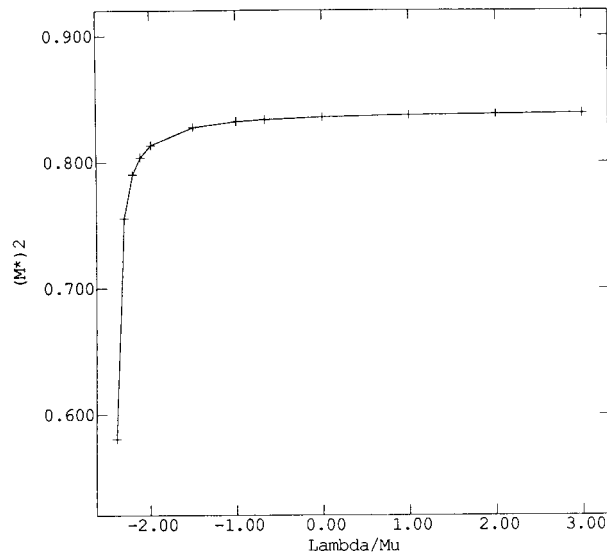


Fig. 4. Computed values of $(M^*)_2$ behind a normal shock with various values of λ/μ and $(M^*)_1 = 1.2$. Stokes' hypothesis ($\lambda/\mu = -2/3$) produces Rankine-Hugoniot shock jumps.

and the local speed of sound, a , is defined in equation (32), it follows that equation (45) after multiplication by a^2 can be rewritten as

$$-\frac{\mu''}{\text{Re}} \left(1 + \frac{\gamma - 1}{\text{Pr}''} \right) \phi_x \phi_{xxx} + \frac{\mu''}{\text{Re}} (\gamma - 1) \left(1 - \frac{1}{\text{Pr}''} \right) \phi_{xx}^2 - \rho \frac{\gamma + 1}{2} (1 - \phi_x^2) \phi_{xx} = 0. \quad (47)$$

Equation (47) was numerically integrated using a fourth-order accurate Runge-Kutta scheme with step size $\Delta x = 0.000001$. The values for Pr and Re were $\text{Pr} = 3/4$ and $\text{Re} = 10^5$. Values of the physical properties μ , k , γ and Pr are well-documented in the existing literature, but experimentally obtained values for λ and Pr'' differ by orders of magnitude. For example, Stokes' hypothesis states that the bulk viscosity is zero [$\mu_B = (2/3)\mu + \lambda = 0$]. Hence, $\lambda = -(2/3)\mu$ is the most frequently used value for the secondary viscosity. Nevertheless, from data compiled by Truesdell [12], $\mu_B \approx (2/3)\mu$ for air—suggesting that $\lambda = 0$.

To investigate the effects of different values of λ on the solution of equation (47), several computer runs were performed. When equation (47) is solved using Stokes' hypothesis ($\bar{\lambda}/\bar{\mu} = -2/3$), then the results will match Rankine-Hugoniot shock jumps (Fig. 2). In order to illustrate the influence of secondary viscosity, λ , on the magnitude of the shock jump, a number of numerical tests were performed with various values of $\bar{\lambda}/\bar{\mu}$ and a fixed value of upstream critical Mach number, $(\phi_x)_1 = 1.2$. The results of this comparison (Fig. 3) confirm the intuitive expectation that smaller values of λ cause steepening of the shock wave since $\mu'' = 2\mu + \lambda$ becomes negligible. The PDP equation is capable of producing shock waves of different strengths, where the variation of the critical Mach number $(\phi_x)_2$ downstream of the normal shock caused by different values of the secondary viscosity λ is shown in Fig. 4. Notice that the Rankine-Hugoniot jump condition is obtained when $\bar{\lambda}/\bar{\mu} = -2/3$ and that the isentropic shock jump conditions [11] will be obtained when $\bar{\lambda}/\bar{\mu} \cong -2$, i.e. when $\mu'' \approx 0$. Thus, the PDP equation accepts Rankine-Hugoniot and isentropic shocks as a part of its general solution.

The PDP equation represents an essentially parabolic partial differential equation [3]. Thus, the two- and three-dimensional versions of the PDP equation can be discretized by applying the same difference formulas everywhere [5]. As demonstrated by Chin [5], there is no need for an explicitly added artificial dissipation when integrating the multidimensional V-T equation. On the basis of this, it is anticipated that no artificial viscosity will be needed when integrating the multidimensional PDP equation. The PDP equation can be an invaluable tool for analyzing the artificial dissipation [13] and generating new physically based models [13, 14] for the artificial dissipation used for the integration of the FPE equation.

CONCLUSIONS

A new analytic model was derived that combines mass, momentum and energy conservation in a single PDE equation for nonsteady, irrotational flow of viscous, heat conducting, calorically perfect gases without body forces. The governing equation is a third-order highly nonlinear partial differential equation which accurately predicts strengths and structures of the shock waves. This equation can be used instead of the FPE as a more appropriate model for transonic shocked flow computations and especially for the more appropriate modelling and analysis of numerical dissipation. In addition, it can be used in nonlinear acoustics where it is important to accurately predict the structure and attenuation of sound waves.

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