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Inverse design and active control concepts in strong unsteady heat conduction

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A summary of recent research in the field of inverse design and optimization of coolant flow passages in the internally cooled configurations is presented. The methodology allows design engineers to prescribe desired surface temperature and heat flux distributions and to fix portions of the multiply connected realistically shaped configurations. The shapes of the resulting coolant flow passages can be arbitrarily or circularly shaped with a capability to maintain certain manufacturing geometric constraints. Unsteady cooling of organs and tissues in bioengineering is demonstrated by determining optimal time variation of thermal boundary conditions on the walls of the cooling container while maintaining the geometry and size of the configuration. Another concept suggests that components subjected to strong unsteady cooling or heating can be optimized for the desired time dependent overspecified surface thermal conditions by determining the corresponding instantaneous temperatures of the coolant flow passages. This effect can be achieved by applying optimal control of distributed coolant flow rates in each flow passage.

INTRODUCTION

During the process of designing a realistic, internally cooled configuration, it is essential to guarantee that the configuration will not melt and burn through during the design and expected off-design operating conditions. Even this simple requirement poses a challenge. The present techniques for the design of such configurations often rely on empiricism, locally one-dimensional heat transfer assumptions, and personal experience of the designer. In addition, final designs are derived from the wealth of experimental results for a number of less successful designs suggested on the basis of personal input from the designer. Thus, at the present time human input based on intuition and available experimental data is still a common ingredient for designing locations and sizing of coolant flow passages, even for steady state thermal boundary conditions.

When the thermal boundary conditions start to vary in time i.e., when the entire range of off-design thermal boundary conditions needs to be satisfied, then the existing design methods do not offer viable tools. In the case of internally cooled gas turbine blades the strong, unsteady, blade heating occurs during engine start-up and strong, unsteady, blade cooling can occur during flame-out at high altitudes when cold air suddenly passes through the hot turbine stages. In both cases the strong, transient thermal stresses may easily cause material fracture. In the case of ceramically coated components, the thermal stresses can result in chipping and separation of the coating from the metal. In the case of a composite material the thermal stresses can cause separation of the laminae.

Thus, it would be very desirable to have an accurate analysis tool that can reliably predict unsteady temperature and heat flux fields in realistically shaped configurations. It would be even more desirable to have a reliable and user-friendly design tool that can efficiently determine optimal sizes and locations of all coolant flow passages for a user-specified steady distribution of surface temperatures and heat fluxes. This methodology and the computational results have been presented in a number of recent [1–10] publications. Finally, the ultimate design would be the one which satisfies the desired steady surface temperatures and heat fluxes, and at the same time, accurately satisfies the expected unsteady, off-design surface temperature and heat flux variations in an automatic fashion [9, 10].

The objective of this paper is to present the concepts and the analytic and numerical methodologies that will allow realistic optimized designs of configurations subjected to strong steady and unsteady heat conduction.

INVERSE DESIGN CONCEPTS

In the case of convective heat transfer to a component for which we are trying to design an adequate cooling system, it is possible to separately determine the surface temperature distribution and the surface heat flux distribution numerically, experimentally, or by a combination of both. For example, the designer can specify a desired surface temperature distribution based on the objective of preventing excessive creep and thermal fatigue. This surface temperature distribution can serve as a boundary condition in an external flow field analysis which will compute the corresponding surface heat flux distribution as a byproduct of the flow field computation.

Steady state heat conduction is governed by the Laplace equation and the boundary conditions can be of Dirichlet type (surface temperature specified), of Neuman type (heat fluxes specified normal to the surface) or as Robin mixed type (a linear combination of surface temperature and surface heat flux specified). The domain is usually multiply connected and of a fixed size and shape. The thermal boundary conditions are often such that independent values for temperature and for the heat flux are simultaneously enforced at the same boundary points.

This is a classical example of an overspecified or ill-posed boundary value problem which has no solution since it has too many constraints. Since separate thermal boundary conditions are to be enforced and the shape and size of the cooled (or heated) component is not to be altered in order to preserve the outside flowfield characteristics, there are two possible ways to satisfy the overspecified boundary conditions and the governing Laplace equation. One option is to modify the size and shape of the domain and the other option is to modify the thermal boundary conditions while maintaining fixed geometry.

The first option is to adjust the interior geometry, i.e., to optimize the sizes, shapes, and locations of the coolant flow passages without altering the outside shape of the turbine blade, scram-jet combustor strut, etc. This approach has been pioneered by the author and his team of research assistants [1–7] and it can be termed inverse design using geometry optimization.

The second option is to adjust the thermal boundary conditions [8, 9, 10] on the walls of the coolant flow passages by independently varying, say, the coolant flow rate in each passage. Thus, when a component with coolant flow passages that are geometrically optimized for an overspecified set of steady design thermal boundary conditions starts operating at off-design conditions, the new time-varying thermal boundary conditions could be met by an optimized time variation of coolant flow rates through the passages.

For example, when a turbine rotor with fixed coolant flow passage geometry starts operating at off-design speeds, the flow transition and separation features will change, thus significantly altering the outer surface temperature and heat flux distribution on the blade. If the designer wants to maintain the desired surface temperature distribution, the new heat flux distribution on the surface could be matched by varying the coolant flow rates through individual coolant flow passages. These concepts should offer practical tools for designing optimal cooling/heating system both for steady design operating conditions and for maintaining the desired unsteady thermal boundary conditions over a range of off-design parameters by implementing adaptive optimum control of the coolant flow rates.

INVERSE DESIGN USING GEOMETRY OPTIMIZATION

The unsteady heat conduction is governed by a linear partial differential equation of parabolic type known as the heat equation. It is only in very special cases of extremely high heat fluxes applied over a very short period of time that the appropriate model should be a hyperbolic partial differential equation [11, 12]. Results of the hyperbolic equation model differ from the results of the heat equation model only during the short initial period.

Although the Laplace equation for steady conduction can be integrated using finite difference or finite element techniques, we have decided to use the surface panel or boundary element method (BEM) which is based on an integral formulation [5, 6, 13]. The main advantage of the BEM for linear problems is that

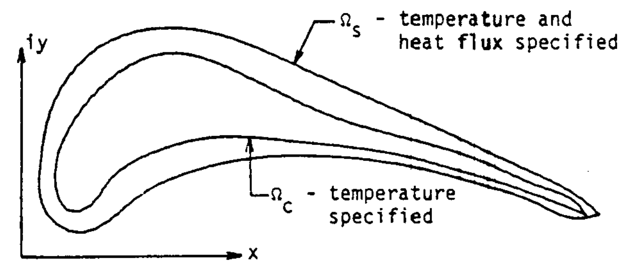

FIG. 1. A typical geometry and thermal boundary conditions.

it requires only a surface grid. This means that only contours of the blade and the coolant flow passages need to be discretized, avoiding the costly task of frequent grid generation for the entire multiply connected domain. The second advantage of the BEM is that the discretization matrix is relatively small and can be inverted directly instead of iteratively. This approach avoids reliability problems related to numerical stability and convergence of iterative schemes. The geometric optimization algorithm for determining sizes, shapes, and locations of the coolant flow passages has been described in earlier works [5–7]. The temperatures on the outside blade surface and on the walls of the holes can be satisfied since they were used as the boundary conditions for the Laplace equation governing steady temperature variation (Fig. 1).

Nevertheless, the resulting surface heat fluxes will not equal to desired specified surface heat fluxes. An error function that is based on a normalized least-squares formulation for the difference between the desired and the computed surface heat fluxes is then formed. It was used in the constrained optimization routine as a forcing function which determined the new updated sizes, shapes and locations of the coolant flow passages so that the difference between the desired and the computed surface heat fluxes is minimized. An example of the ceramically coated turbine blade design history is shown in Figure 2. In the case of a scram-jet combustor strut, the desired thermal boundary conditions are shown in Figure 3. The convergence history for the computed hole sizes and locations is presented in Figure 4.

Similar reasoning can be used to perform unsteady heat equation integration using BEM with an appropriate unsteady fundamental solution [9, 10, 13, 14]. Actually, the unsteady BEM that we chose uses a time linearized technique where the transient temperature solution is obtained over the entire domain and then advanced in time using simple time-stepping integration [13, 14]. This approach requires that the computational grid be generated over the entire domain (Figs. 5–7). We have successfully accomplished this task with an automatic nonstructured grid generation code for multiply connected two-dimensional regions [15]. It is possible to use yet another unsteady BEM method which does not require a grid over the entire domain since the time dependency is analytically built into the fundamental solution [16]. Nevertheless, the computational effort with this method grows rapidly with time since the size of the coefficient matrix is increasing with time.

OPTIMIZATION

The optimization process rests on the definition of a cost function (error measure) and a method for finding a minimum of the cost function. This could easily pose a serious challenge, especially when the number of variables that need to be optimized is large and the cost function has a number of local minima. Theoretical and practical aspects of optimization meth-

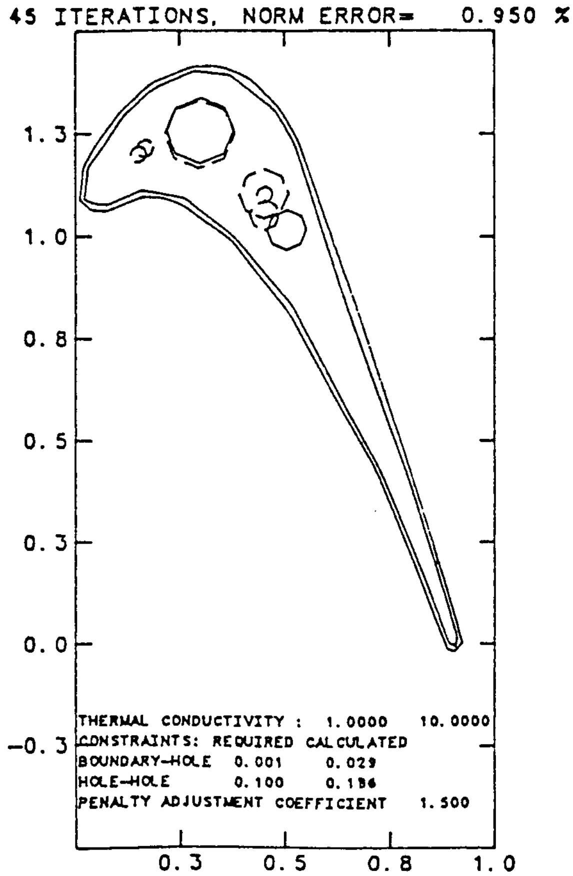

FIG. 2. Inverse design of two circular coolant flow passages in a ceramically coated turbine blade.

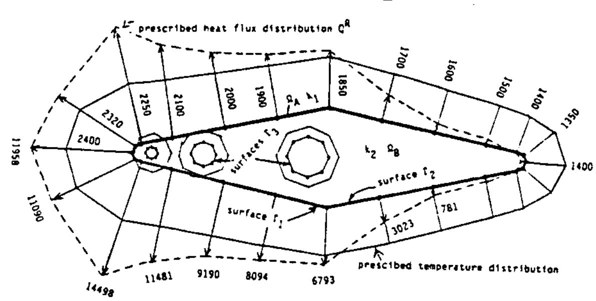

FIG. 3. A sketch of a ceramically coated scram-jet combustor strut with specified surface temperatures and heat fluxes.

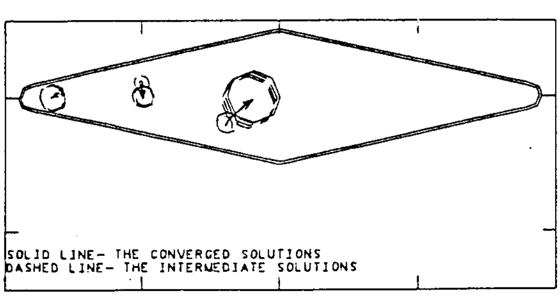

FIG. 4. Inverse design of three circular coolant flow passages in the ceramically coated scram-jet combustor strut (Fig 3).

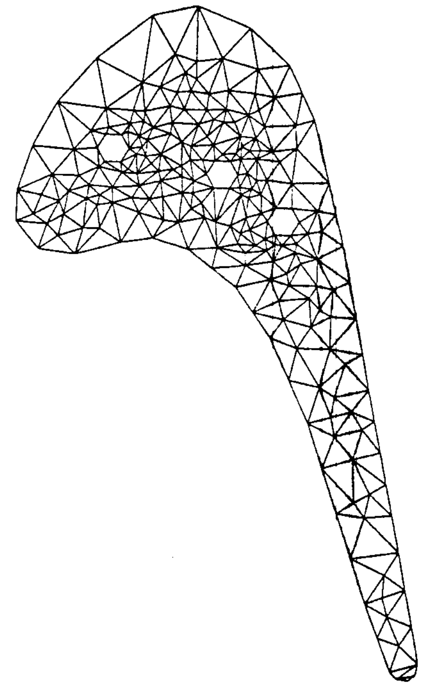

FIG. 5. Non-structured computational grid for a turbine blade with three circular coolant flow passages (initial configuration).

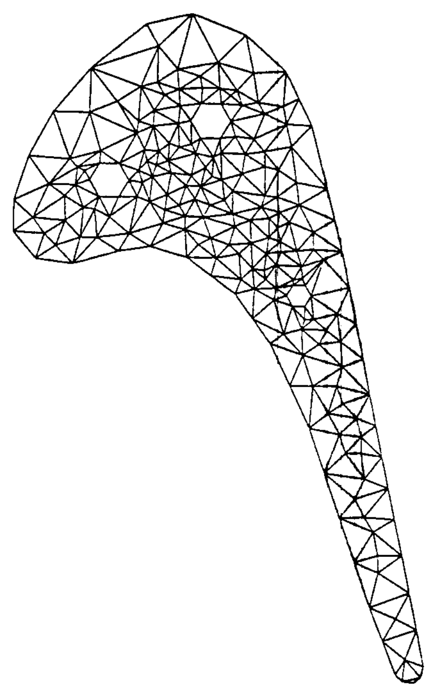

FIG. 6. Non-structured computational grid for a turbine blade with three circular coolant flow passages (optimized configuration).

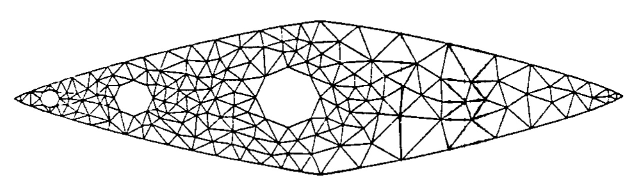

FIG. 7. Non-structured computational grid for a scram-jet strut.

odologies can be found in a number of textbooks [17–19]. In some of our earlier work, the shape of each of the coolant flow passages (holes) was allowed to change during optimization [1–4]. Each of the N coolant flow passage contours was discretized with M surface panels. Since each panel is defined by N pairs of (x, y) coordinates of its edge points, we had to optimize $N \times M \times 2$ variables. This was a computationally intensive [1] task, since we also wanted the shapes to be smooth (Fig. 8).

In general, the resulting shapes of the coolant flow passages were not suitable for manufacturing, which prompted us to restrict shapes of the coolant flow passages to circles. If the designer decides to work with N circular holes, then he will have to optimize N radii and N pairs of (x, y) coordinates defining the centers of these circles ($N \times 3$ variables). Optimization methods that we have been using in the past are all based on gradient evaluation and separate, local, one-dimensional searching in the direction of each subgradient. The methods of Fletcher–Reeves and Davidson–Fletcher–Powell are common [17–19].

In gradient based methods, it is very important to find the appropriate magnitude of the step size that needs to be taken in the generalized search direction. The usual name for this term is the search direction parameter or search step size. If this parameter is too small, then an excessively large number of cost function evaluations will need to be performed, and the optimization process will become uneconomical. In addition, a very small step size can easily lead the optimization process to a local minimum instead of a global minimum. On the other hand, if the search parameter is chosen to be too large, then narrow global minima can easily be missed, in which case the optimization process rapidly diverges.

The author of this paper has recently suggested a simple approach to the problem of finding global minima [9, 10, 12]. The entire range of possible search parameters can be normalized so that they vary between 0 and 1. This approach suggests that, say, 10 analysis runs should be performed on 10 perturbed configurations where each of them was obtained with a different value of the scaled search parameter.

First, we evaluate the partial derivative of the global cost function with respect to the variable that we are trying to optimize. This should be performed for each of the 10 solutions. Then we fit a very smooth spline curve through the 10 values of partial gradients. For this purpose we have found that exponential spline fitting [20] provides excellent results. At the same time, these 10 values should be interpolated at, say, 1000 equidistantly spaced values of the search parameter, Then, a simple search should be performed to find which of the thousand interpolated values of the partial gradient has the minimal value. The corresponding value for the normalized search parameter is then used to update the optimized variable. This spline fitting and interpolation procedure should be repeated with all remaining variables that need to be optimized, thus completing the first cycle.

After all the optimized variables have been updated, the next analysis run is performed on this modified configuration. The cost function for this run and separate gradients with respect to each variable are obtained. Then, these gradients should be inserted at the locations of the search parameters that were obtained during the first cycle. Thus, there will now be 11 points instead of 10 for each optimized variable. The exponential spline is fitted through these 11 points and interpolated at a 1000 equidistantly spaced values of the normalized search parameter. The minimal value among the 1000 values is found and the corresponding one-dimensional search parameter is used to update the optimized variable. This is repeated with all other variables that need to be optimized, thus completing the second cycle. The process should be repeated by adding one additional point to the spline fitting after each cycle until a desired convergence tolerance is achieved.

This algorithm proved to be reliable and straightforward to implement. In the case of optimized cooling of transplant organs [9, 10], which uses a similar approach, all standard techniques for determining the search parameter failed until we started using the spline fitting and interpolation algorithm. This approach also provides monotonic convergence for the global optimization algorithm.

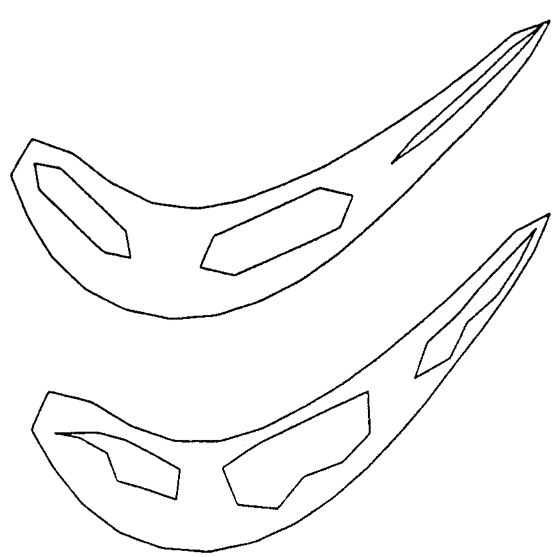

FIG. 8. Initial and optimized configurations of arbitrarily shaped coolant flow passages in a turbine blade.

OPTIMUM CONTROL OF LOCAL COOLING RATES

This concept will be demonstrated by using an example from bioengineering [9, 10]. When preserving living human tissues (kidney, heart, liver, embryo, bone, skin, spleen, semen, etc) for the purpose of performing transplant surgery, the tissue is cooled in a special cryoprotective liquid to a prescribed low temperature and kept at this temperature until used. During the cooling process there should be an optimal cooling rate for each particular type of tissue of an organ in order to maximize the survivability of the living cells and reduce the problem of future rejection by the organ recipient's body. Once the optimal cooling rates for each type of tissue have been experimentally determined, they can be enforced in a number of ways.

One method that offers a practical solution is to determine the proper surface thermal conditions of the container in which the cryoprotective liquid and the organ are located so that the optimal local cooling rates are achieved at each instant of time at every point in the organ. This represents an entirely new approach [9, 10] to living tissue banking that offers the highest probability of minimizing the damage to the living tissue.

While the size and geometry of the transplant organ and the container remain fixed, the wall temperature of the cooling container can be continuously adjusted in order to maintain the specified local cooling rates throughout the organ. To implement this in time, the circumferential variation of temperature on the container walls can be approximated with, say, Chebyshev polynomials in terms of the scaled circumferential angles [10]. The coefficients of these polynomials could be adjusted iteratively in order to maintain the desired local cooling rates inside the organ. The initial values of the coefficients of the polynomials are specified and the transient temperature distribution is computed at every point of the organ. From this, the local cooling rates are computed at each point of the organ.

A normalized error function can then be formed as a sum of the least squares of deviations of the computed and the specified optimal local cooling rates. A nonlinear constrained function minimization method [10, 17–19] is then used to compute new values of the polynomial coefficients. That is, the new temperature distribution on the container walls is determined in order to reduce the error function at the next time step during the cooling process. Thus, the desired optimal local cooling rates are achieved throughout the organ by determining the proper instantaneous values of the coefficients of the polynomial representing the temperature variation on the surface of the cooling container.

To demonstrate a practical application of the optimization design process, an actual canine kidney (Fig. 9) was approximated by three multiply connected regions (Fig. 10). The outer annulus simulated the cryoprotective fluid [Fig. 10(a)] and the two inner regions [Figs. 10(b) (c)] simulated the two distinct tissues of an actual kidney. The diffusivities used (Fig. 9) were $\alpha_1 = 0.00145$, $\alpha_2 = 0.0169$, and $\alpha_3 = 0.0255$. Linear spatial (BEM) interpolation and constant time interpolation were used for this example. The initial temperature of the configuration was 305K and the optimal cooling rate was enforced to be $\partial T/\partial t = -2.5$K/min within every triangular element forming the domain approximating the two types of kidney tissue. A sixth-order Chebyshev polynomial in terms of the circumferential angle was used to represent the variable boundary temperature on the surface of the container.

Due to the low diffusivity of the cryoprotective fluid, the cooling rate was not optimized during the first five minutes of the problem (Fig. 11). The relative average error of the cooling rate throughout the kidney with the initial container wall temperature kept fixed was 11.7% after 10 min (Fig. 12). In contrast, an optimization of the container wall temperatures performed after 5 min resulted in the isotherms shown in Figure 13. The relative error of the cooling rate for this case after optimization of the boundary temperatures was 0.26%. The isotherms after 15 and 20 minutes are shown in Figures 14 and 15, respectively. The relative cooling rate errors for these two time periods were held to less than one percent. Circumferential variation of optimized container wall temperatures (Fig. 16) and heat fluxes (Fig. 17) indicate the potential of this method and the feasibility of the enforcement of optimized local cooling rates. As a byproduct, this methodology can be applied to design improved fully controlled molds for fast solidification of metallic and ceramic materials which is of considerable interest to a variety of industrial processes. In addition, it can also be applied to optimized thermal control of large scale facilities as well as household refrigerators for freezing of meat, vegetable, and fruit.

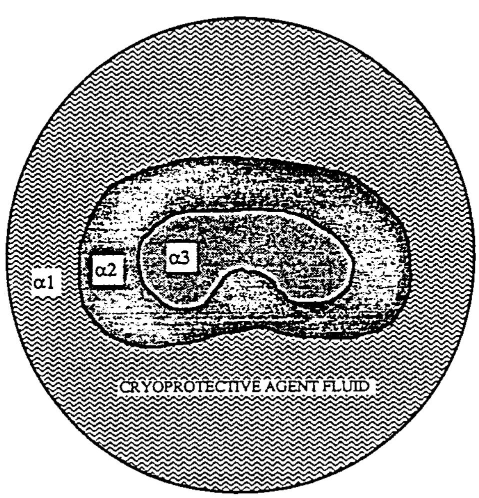

FIG. 9. A sketch of a two-tissue kidney submerged in a cryoprotective fluid in a circular cooling container.

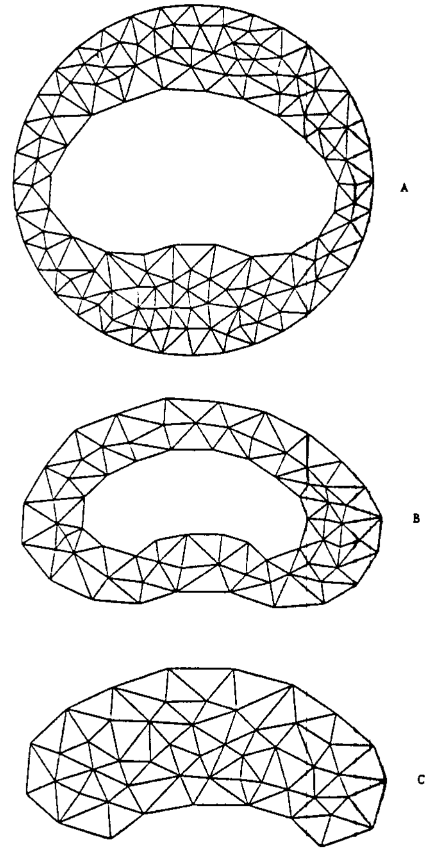

FIG. 10. Discretized regions of the kidney cooling environment: a) cryoprotective fluid, b) outer kidney tissue, c) inner kidney tissue.

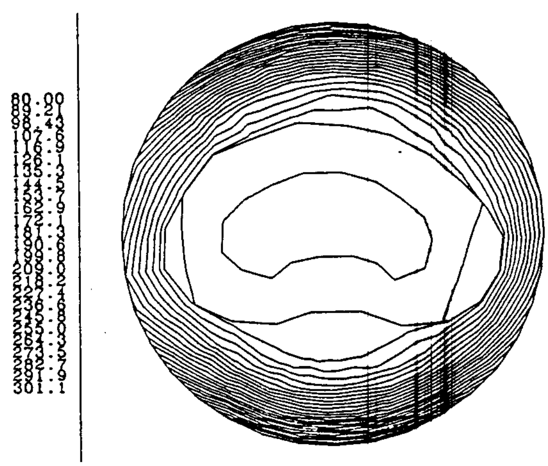

FIG. 11. Isotherms for the kidney cooling after 5 minutes.

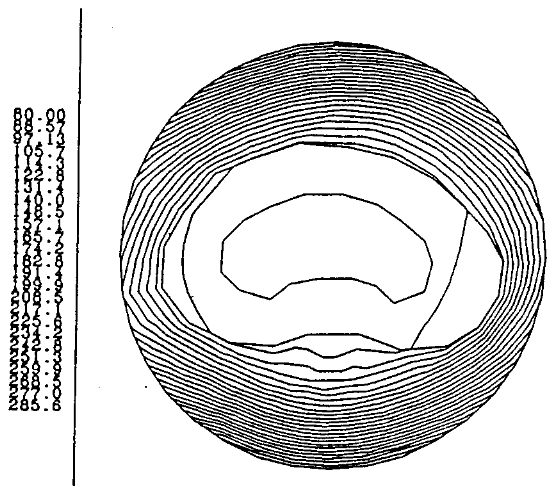

FIG. 12. Isotherms for the kidney cooling after 10 minutes (no optimization).

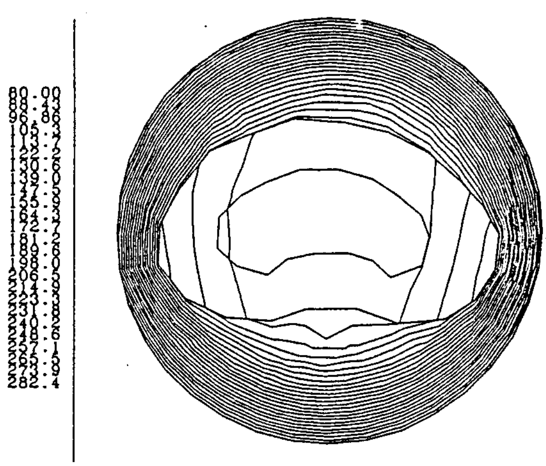

FIG. 13. Isotherms for the kidney cooling after 10 minutes with optimization.

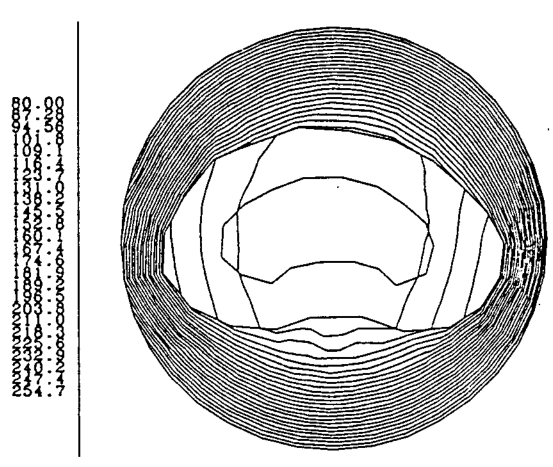

FIG. 14. Isotherms for the kidney cooling after 15 minutes with optimization.

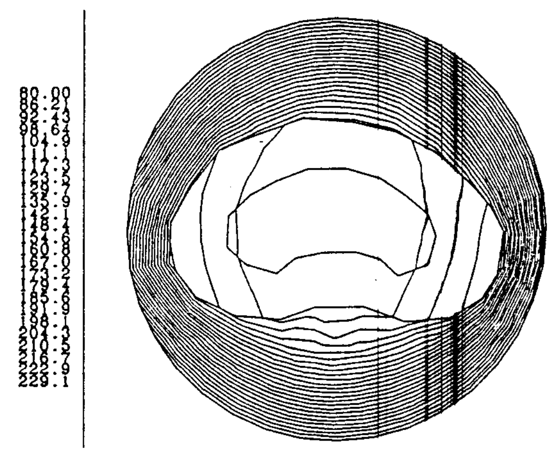

FIG. 15. Isotherms for the kidney cooling after 20 minutes with optimization.

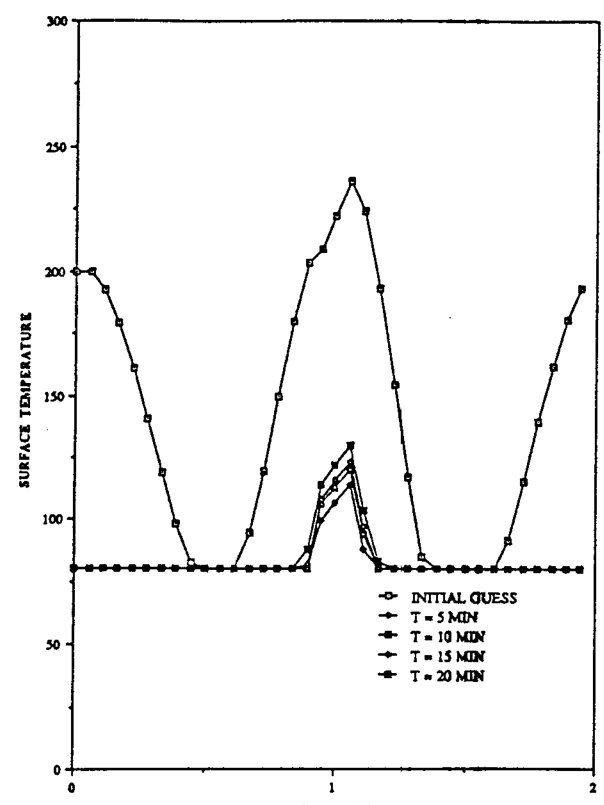

FIG. 16. Circumferential distribution of the optimized container wall temperatures.

OPTIMUM CONTROL OF COOLANT FLOW RATES

Imagine that it is possible to vary the temperatures or heat fluxes of each coolant flow passage independently during the fast heating or cooling of the object. If the configuration was previously optimized for the specified outside surface steady temperature and heat flux distribution, then it should be possible to maintain almost the same surface outside temperatures even when the heat fluxes on the outside surface start changing. This could be achieved by varying coolant flow rates independently inside each hole since the geometry of the holes cannot be changed. Specifically, at each instant of time the outside

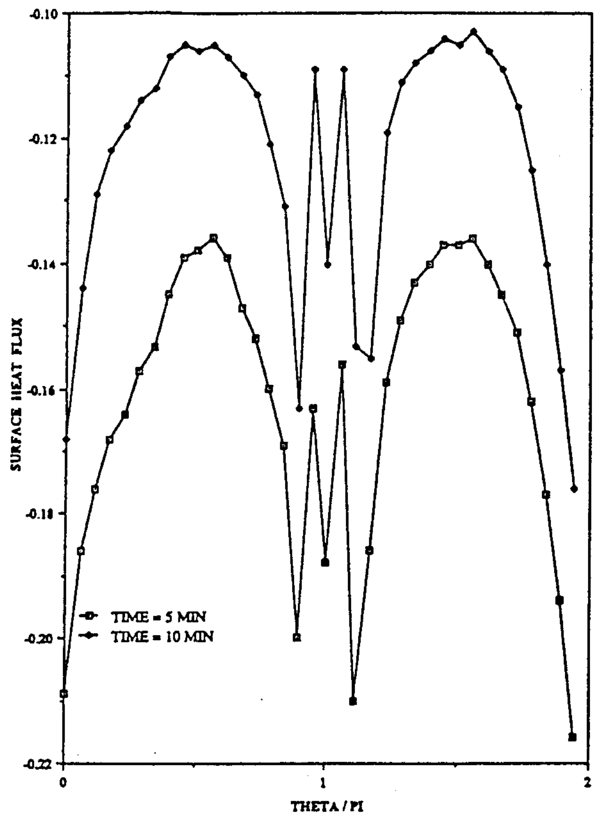

FIG. 17. Circumferential distribution of the optimized container wall heat fluxes.

surface temperature and heat flux values are known. We also know the exact fixed geometry of the holes and the coolant temperatures, from which we can estimate temperatures of the coolant passage walls. Then, the optimizer should determine how much to alter the temperatures on the walls of each circular hole at each instant of time so that certain constraints are satisfied. These constraints can be, say, the maximum local time variation of the temperature at any point within the configuration [9, 10], the maximum local spatial variation of temperature at any point inside the configuration during the entire unsteady process, etc.

The instantaneous values of coolant passage wall temperatures should be used as the thermal boundary conditions in a time-accurate Navier–Stokes analysis code predicting coolant flow field and thermal field characteristics. By integrating the resultant velocity field/density field, an instantaneous mass flow rate can be obtained for each coolant flow passage. These optimum values of the distributed mass flow rates must vary with time to insure the desired temperature (or heat flux) variations on the outer surface of the object.

It should be pointed out that this concept of optimum distributed control can also be used to achieve and maintain certain flow field properties. The viscosity of liquids decreases with rise in temperature and the viscosity of gases increases with rise in temperature. Consequently, the thickness of a boundary layer and its tendency to separate can be influenced by distributed surface heat transfer. Moreover, if the surface thermal boundary conditions are automatically varied in time, we should be able to influence the unsteady flow fields. Thus, the optimum distributed control of coolant flow rates could also be used to minimize and possibly even suppress unsteady

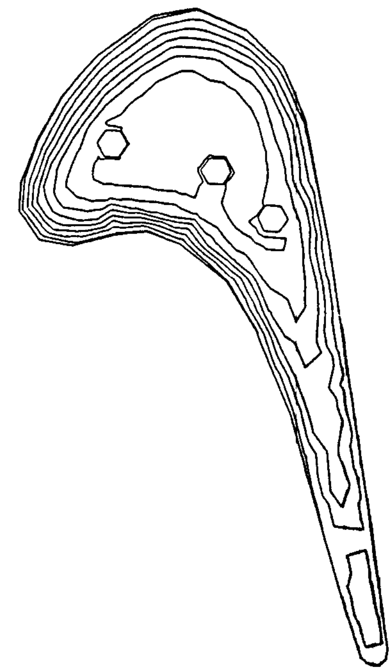

FIG. 18. Isotherms in an unoptimized turbine blade after 4 seconds.

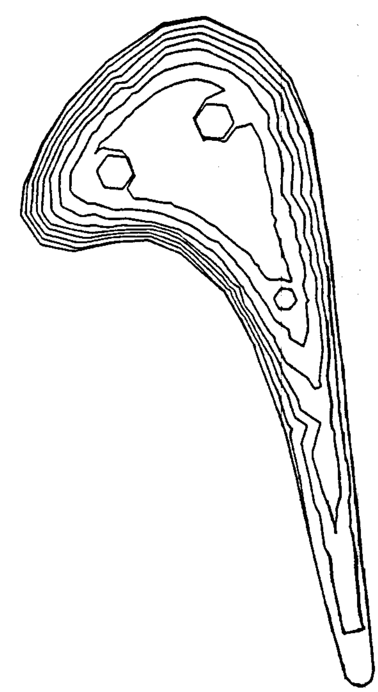

FIG. 19. Isotherms in an optimized turbine blade after 4 seconds.

rotating stall in turbomachinery stages. An even more important application of this concept is possible in the case of actively cooled scram-jet combustor struts [7, 21]. Extreme thermal stresses that exist in the strut panels cause slight buckling of the panels. Nevertheless, slight changes in the shape and size of the spaces separating neighboring struts can cause significant change in the mass flow rate of air passing through. This in turn can dramatically change the flow field properties

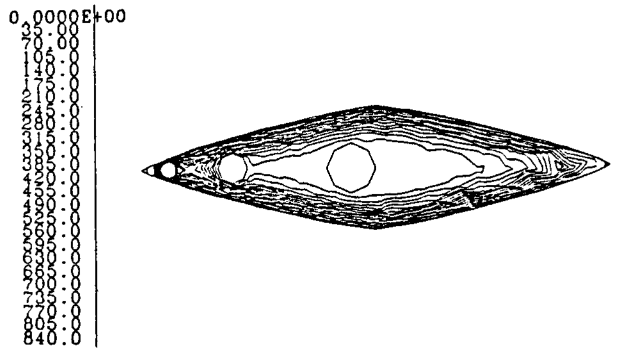

FIG. 20. Isotherms in an optimized scram-jet strut after 2 seconds.

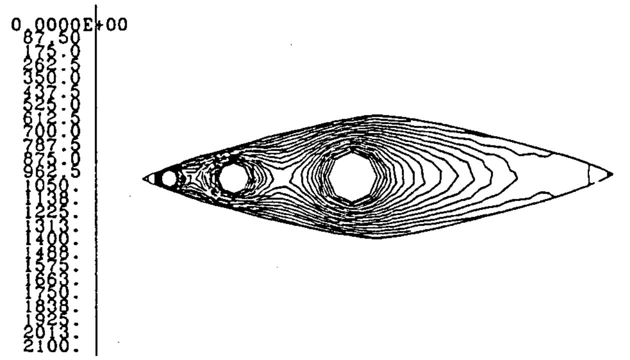

FIG. 21. Isotherms in an optimized scram-jet strut after 30 seconds (steady).

and cause either a premature combustion or flame blow-out. At hypersonic speeds for which the scram-jets are being designed, either problem can result in the loss of a vehicle since the time for a semi-infinite solid to reach ablation temperature is only a fraction of a second [22].

We ran the unsteady BEM analysis code on the turbine blade with both unoptimized and optimized configurations of the circular holes. The nonstructured computational grid used in both unsteady analysis runs is shown in Figures 5 and 6. The initial temperature in both cases was 200°C. The temperatures on the blade surface were then linearly increased over a period of ten seconds, when they reached the steady values. The computed isotherms for the unoptimized configuration are shown in Figure 18, and the sequence of isotherms for the previously optimized configuration are shown in Figure 19. During this fast heating process, the temperatures of the walls of each circular hole were kept fixed at 200°C. Similar computations were performed with a scram-jet strut (Figs. 20 and 21). Thus, it is desirable to develop such a methodology that will maintain isothermal strut surface conditions at various angles of attack and various flight Mach numbers. The suggested optimum distributed control of coolant flow rates inside each strut could be a viable method for the design of actively cooled panels.

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