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THEORY OF UNSTEADY COMPRESSIBLE IRROTATIONAL FLOWS INCLUDING HEAT CONDUCTIVITY AND LONGITUDINAL VISCOSITY

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ABSTRACT

A new exact analytical model was derived for the irrotational, unsteady flows of compressible fluids when effects of heat conductivity and molecular viscosity are allowed. This new model satisfies conservation of mass, momentum, and energy exactly. In addition, it satisfies physical irrotationality conditions. Compared to the classical small perturbation Viscous-Transonic (V-T) equation, the new Physically Dissipative Potential (PDP) equation contains a number of additional terms that are highly nonlinear. The new model is derived in a general vector operator form and in a scalar canonical form.

INTRODUCTION

An exact analytical model for nondissipative irrotational inviscid heat nonconducting compressible fluid flow is the Full Potential Equation (FPE). During the late forties, Cole [1] derived a new analytic model for potential, steady, two-dimensional flows by partially incorporating heat conductivity and secondary viscosity effects. During that period linearization methods based on small perturbation theory were very popular. This is a possible reason why this original Viscous-Transonic (V-T) equation retains only the most essential physical nonlinearities. Actually, as clearly stated in the works of Sichel [2,3], V-T equation represents a combination of the classical transonic small perturbation equation which contains the most essential nonlinearities of inviscid flows and the Burgers equation which contains the most essential linear dissipation effects. Ryzhov [4] used physical arguments to justify small perturbation linearization processes utilized in the derivation of V-T equations for planar and axisymmetric flows. These authors succeeded also in obtaining analytic solutions for V-T equation governing transonic flows about thin airfoils, thin projectiles and through shock waves. Chin [5] successfully integrated the V-T equation numerically for a steady two-dimensional transonic flow around an airfoil.

The objective of this work is to derive an exact Physically Dissipative Potential (PDP) equation without resorting to linearizations. Thus, our intention is to create a physical model that is more complete than the classical V-T equation and the FPE and which is based on a single dependent variable.

CONSERVATION LAWS

Equation of state for a thermally perfect gas

$$p = \rho R T \tag{1}$$

links thermodynamic static pressure, p, density, ρ, and absolute temperature, T. From Eq. 1 it follows that

$$\ln p = \ln \rho + \ln R + \ln T \tag{2}$$

For a perfect gas R = constant. Then, total time derivative of the above equation becomes

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{T}\frac{DT}{Dt} \tag{3}$$

Mass conservation can be expressed as

$$\rho(\nabla \cdot \vec{V}) = -\frac{D\rho}{Dt} + \dot{m} \tag{4}$$

where $\dot{m}$ designates the rate of generation of mass per unit time per unit volume and $\vec{V}$ is the local fluid velocity vector. After introduction of Eq. 3 and Eq. 1, the mass conservation equation becomes

$$\rho(\nabla \cdot \vec{V}) = \frac{\rho}{T}\frac{DT}{Dt} - \frac{1}{RT}\frac{Dp}{Dt} + \dot{m} \tag{5}$$

Energy conservation can be expressed as

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} = \Phi + \nabla\cdot(k\nabla T) - \nabla\cdot\dot{\vec{q}}_r + \dot{Q} - \dot{m}\left(u + \frac{p}{\rho} - \frac{\vec{V}\cdot\vec{V}}{2}\right) \tag{6}$$

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where u is the specific internal energy, h is the specific enthalpy, k is the heat conduction coefficient of the fluid assuming Fourier's law, Φ is the viscous dissipation function, $\dot{\vec q}_r$ is the time rate of radiation heat flux vector, $\dot Q$ is the time rate of internal heat generation.

The viscous dissipation function Φ is defined in its vector operator form as

$$\Phi = 2\mu\{\nabla\cdot[(\vec V\cdot\nabla)\vec V] + \frac{1}{2}(\nabla\times\vec V)^2 - (\vec V\cdot\nabla)(\nabla\cdot\vec V)\}$$

$$+ \lambda(\nabla\cdot\vec V)^2 \tag{7}$$

where μ is the shear viscosity coefficient and λ is the secondary viscosity coefficient. For calorically perfect gases

$$h = C_pT = u + \frac{p}{\rho} \tag{8}$$

where the specific heat at constant pressure, C_p, is constant. Therefore, Eq. 6 divided with C_pT can be rewritten as

$$\frac{\rho}{T}\frac{DT}{Dt} = \frac{1}{C_pT}\frac{Dp}{Dt} + \frac{1}{C_pT}[\Phi + \nabla\cdot(k\nabla T)$$

$$- \nabla\cdot\dot{\vec q}_r + \dot Q - \dot m(h - \frac{\vec V\cdot\vec V}{2})] \tag{9}$$

Substitution of Eq. 9 (energy conservation) in Eq. 5 (mass conservation) results in

$$\rho(\nabla\cdot\vec V) = (\frac{1}{C_pT} - \frac{1}{RT})\frac{Dp}{Dt} + \frac{1}{C_pT}[\Phi+\nabla\cdot(k\nabla T)$$

$$- \nabla\cdot\dot{\vec q}_r + \dot Q - \dot m(h - \frac{\vec V\cdot\vec V}{2})]+\dot m \tag{10}$$

Note that

$$(\frac{1}{C_pT} - \frac{1}{RT}) = (\frac{\gamma-1}{\gamma RT} - \frac{\gamma}{\gamma RT}) = -\frac{1}{a^2} \tag{11}$$

where $\gamma = C_p/C_v$ and a is the local isentropic speed of sound. Then, Eq. 10 becomes

$$\rho(\nabla\cdot\vec V) = -\frac{1}{a^2}\frac{Dp}{Dt} + \frac{\gamma-1}{a^2}[\Phi + \nabla\cdot(k\nabla T)$$

$$- \nabla\cdot\dot{\vec q}_r + \dot Q - \dot m(h - \frac{\vec V\cdot\vec V}{2})]+\dot m \tag{12}$$

Momentum conservation can be expressed as

$$\nabla p = \rho\vec b - \rho\frac{D\vec V}{Dt} - \dot m\vec V + \{2\{\nabla(\mu\nabla\rangle\vec V\big)$$

$$-\nabla\times[\mu(\nabla\times\vec V] + \nabla[\lambda(\nabla\cdot\vec V)]\} \tag{13}$$

where $\vec b$ is the body force per unit mass.

Pre-multiplying Eq. 13 with $\vec V$ and using the vector identity

$$(\vec V\cdot\nabla)\vec V = \nabla(\frac{\vec V\cdot\vec V}{2}) - \vec V\times(\nabla\times\vec V) \tag{14}$$

it follows that the total differential

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + (\vec V\cdot\nabla)p \tag{15}$$

can be written as

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \rho[\vec V\cdot\vec b - \vec V\cdot\frac{\partial\vec V}{\partial t} - (\vec V\cdot\nabla)(\frac{\vec V\cdot\vec V}{2})]$$

$$+ \vec V\cdot\{2\{\nabla(\mu\nabla\rangle\vec V\big) - \nabla\times[\mu(\nabla\times\vec V)] + \nabla[\lambda(\nabla\cdot\vec V)]\}$$

$$+\rho\vec V\cdot(\vec V\times(\nabla\times\vec V))\} - \dot m\,\vec V\cdot\vec V \tag{16}$$

Hence, mass conservation (Eq. 12) which already includes energy conservation, becomes after inclusion of momentum conservation (Eq. 16) the following expression

$$\rho\ \{\frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{a^2}\vec V\cdot\frac{\partial\vec V}{\partial t} + [(\nabla\cdot\vec V) - \frac{1}{a^2}(\vec V\cdot\nabla)$$

$$(\frac{\vec V\cdot\vec V}{2})]\} = \frac{-1}{a^2}\{\rho[\vec V\cdot\vec b + \vec V\cdot(\vec V\times(\nabla\times\vec V))]+$$

$$\vec V\cdot[2\{\nabla(\mu\nabla\rangle\vec V\big) - \nabla\times(\mu(\nabla\times\vec V)) + \nabla(\lambda(\nabla\cdot\vec V))]$$

$$- (\gamma-1)[\Phi+\nabla\cdot(k\nabla T) - \nabla\cdot\dot{\vec q}_r + \dot Q - \dot m(h - \frac{\vec V\cdot\vec V}{2})]\}$$

$$+ \dot m(1+M^2) \tag{17}$$

Here, the local Mach number is defined as $M = |\vec V|/a$. Since $a^2=\gamma RT$, note that

$$\dot m\ [-\frac{\gamma-1}{a^2}(h - \frac{\vec V\cdot\vec V}{2}) + (1+M^2)] =$$

$$\dot m\ [-\frac{\gamma-1}{\gamma RT}C_pT + \frac{\gamma-1}{2}M^2 + 1 + M^2] \tag{18}$$

Also, since

$$C_p = \frac{\gamma R}{\gamma-1} \tag{19}$$

it follows that Eq. 18 can be rewritten as

$$\frac{\gamma-1}{a^2}[-\dot m(h - \frac{\vec V\cdot\vec V}{2})] + \dot m(1+M^2) = \dot m M^2(\frac{\gamma+1}{2}) \tag{20}$$

Hence, mass conservation (Eq. 17) can be written as

$$\rho\left\{\frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{a^2}\vec{V}\cdot\frac{\partial \vec{V}}{\partial t} + \left[(\nabla\cdot\vec{V} - \frac{1}{a^2}(\vec{V}\cdot\nabla)\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\right]\right\} = \frac{-1}{a^2}\left\{\rho[\vec{V}\cdot\vec{b} + \vec{V}\cdot(\vec{V}\times(\nabla\times\vec{V}))]\right.$$

$$+ \vec{V}\cdot[2\nabla(\mu\nabla\cdot\vec{V}) - \nabla\times(\mu(\nabla\times\vec{V})) + \nabla(\lambda(\nabla\cdot\vec{V}))]\}$$

$$+ \frac{\gamma-1}{a^2}\left\{\Phi+\nabla\cdot(k\nabla T) - \nabla\cdot\dot{\vec{q}}_r + \dot{Q}]\right\} + \dot{m}M^2\left(\frac{\gamma+1}{2}\right) \tag{21}$$

Notice that Eq. 21 is an exact formula for mass conservation that also implicity satisfies exact momentum and energy conservation equations for a calorically perfect gas.

IRROTATIONALITY CONDITION

Gibbs relation expressed in its vector operator form as

$$T\nabla s - \nabla h = -\frac{1}{\rho}\nabla p \tag{22}$$

can be expanded by adding $\nabla(\frac{\vec{V}\cdot\vec{V}}{2})$ to both sides, that is,

$$T\nabla s - \nabla(h + \frac{\vec{V}\cdot\vec{V}}{2}) = -(\frac{1}{\rho}\nabla p + \nabla(\frac{\vec{V}\cdot\vec{V}}{2})) \tag{23}$$

Introduction of Eq. 14 in Eq. 23 results in an equation similar to the Crocco-Wazsonyi [6] equation.

$$T\nabla s - \nabla h_o = -\vec{V}\times(\nabla\times\vec{V}) + \frac{\partial\vec{V}}{\partial t} - \vec{b}$$

$$-\frac{1}{\rho}\{2\nabla(\mu\nabla\cdot\vec{V}) - \nabla\times[\mu(\nabla\times\vec{V})]$$

$$+ \nabla[\lambda(\nabla\cdot\vec{V})] - \dot{m}\vec{V}\} \tag{24}$$

This equation is valid for unsteady flow of a calorically perfect compressible, heat conducting, viscous fluid under the influence of body forces and allowing for mass sources and sinks.

Here, $h_o = h + \frac{\vec{V}\cdot\vec{V}}{2}$ is the stagnation enthalpy per unit mass. Assumption that the stagnation quantities are constant implies that the flow is homoenergetic ($\nabla h_o = 0$).

If body forces and mass generation are neglected and the flow is assumed to be irrotational ($\nabla\times\vec{V} = 0$), then Eq. 24 becomes

$$T\nabla s = \frac{\partial\vec{V}}{\partial t} - \frac{\mu''}{\rho}\nabla(\nabla\cdot\vec{V}) \tag{25}$$

where μ'' is the longitudinal [3] viscosity coefficient $\mu'' = 2\mu + \lambda$.

This means that the flow can be potential ($\vec{V} = \nabla\phi$), although non-isentropic and that the entire flow field can be described with a single variable called the velocity potential function ϕ. This general concept of non-isentropic potential flows was clearly described by Klopfer and Nixon [7]. Actually, "the assumption of irrotational flow, which is a key step in the present development, cannot be rigorously justified a priori" [3]. Thus, the following derivation is "based on the concept of a fluid which has only compression viscosity so that it can still slip over the airfoil surface as in inviscid flow" [3].

THE PHYSICALLY DISSIPATIVE POTENTIAL FLOW EQUATION

If coefficients μ, λ, and k are assumed constant, then neglecting body forces and mass generation, momentum conservation (Eq. 13) becomes

$$\frac{\partial\vec{V}}{\partial t} + \nabla(\frac{\vec{V}\cdot\vec{V}}{2}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\{2\mu\nabla(\nabla\cdot\vec{V})$$

$$-\mu[\nabla\times(\nabla\times\vec{V})] + \nabla[\lambda(\nabla\cdot\vec{V})]\} \tag{26}$$

Since the flow is assumed to be irrotational ($\nabla\times\vec{V} = 0$), using the vector identity

$$\nabla^2\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla\times(\nabla\times\vec{V}) \tag{27}$$

in Eq. 26 results in

$$\frac{\partial\vec{V}}{\partial t} + \nabla(\frac{\vec{V}\cdot\vec{V}}{2}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla[2\mu(\nabla\cdot\vec{V}) + \lambda(\nabla\cdot\vec{V})] \tag{28}$$

Hence, with the assumption of a potential flow ($\vec{V} = \nabla\phi$) this equation becomes

$$-\nabla[\frac{\partial\phi}{\partial t} + \frac{\nabla\phi\cdot\nabla\phi}{2}] = \frac{1}{\rho}\nabla[p - (2\mu+\lambda)\nabla^2\phi] \tag{29}$$

After taking partial derivative of both sides with respect to time and dividing both sides with a^2 it follows that

$$-\frac{1}{a^2}[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(\frac{\nabla\phi\cdot\nabla\phi}{2})] =$$

$$\frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{\rho a^2}\frac{\partial}{\partial t}[(2\mu+\lambda)\nabla^2\phi] \tag{30}$$

Finally,

$$\frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{a^2}\vec{V}\cdot\frac{\partial \vec{V}}{\partial t} = -\frac{1}{a^2}\left[\frac{\partial^2 \phi}{\partial t^2} + \nabla\phi\cdot\frac{\partial}{\partial t}(\nabla\phi) + \frac{\partial}{\partial t}\frac{(\nabla\phi\cdot\nabla\phi)}{2}\right] + \frac{1}{\rho a^2}\frac{\partial}{\partial t}\left[(2\mu+\lambda)\nabla^2\phi\right] \qquad (31)$$

should be substituted in Eq. 21 when the flow is irrotational.

Hence, the general vector operator form of mass conservation (Eq. 21) in the case of an irrotational flow of a calorically perfect gas allowing for heat conduction, shear viscosity, and secondary viscosity (but neglecting body forces, radiation heat transfer, and mass and heat sources) becomes

$$\rho\left\{\nabla^2\phi - \frac{1}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial(\nabla\phi\cdot\nabla\phi)}{\partial t} + (\nabla\phi\cdot\nabla)\left(\frac{\nabla\phi\cdot\nabla\phi}{2}\right)\right]\right\}$$

$$= -\frac{1}{a^2}\nabla\phi\cdot\{\nabla[(2\mu+\lambda)\nabla^2\phi]\}$$

$$+ \frac{\gamma-1}{a^2}\{2\mu[\nabla\cdot((\nabla\phi\cdot\nabla)\nabla\phi) - \nabla\phi\cdot\nabla(\nabla\cdot\nabla\phi)] + \lambda(\nabla^2\phi)^2\}$$

$$+ \frac{\gamma-1}{a^2}\nabla\cdot(k\nabla T) - \frac{1}{a^2}(2\mu+\lambda)\frac{\partial}{\partial t}\nabla^2\phi \qquad (32)$$

CANONICAL FORM

The above equation can be expressed in a locally streamline aligned [8] Cartesian coordinate system (s, m, n). Here, s is the streamline direction and n and m form a plane perpendicular locally to the streamline. The velocity components normal to the streamlines are zero ($\phi_m = \phi_n = 0$) by definition and $a^2 = \gamma RT$. By introducing the coefficient of longitudinal [9] viscosity $\mu'' = 2\mu + \lambda$, it follows that Eq. 32 transforms to

$$\rho\left\{(\phi_{ss}+\phi_{mm}+\phi_{nn}) - \frac{1}{a^2}(\phi_{tt} + 2\phi_s\phi_{st}) - \frac{1}{a^2}\left(\phi_s\frac{\partial}{\partial s} + \phi_m\frac{\partial}{\partial m} + \phi_n\frac{\partial}{\partial n}\right)\left(\frac{\phi_s^2+\phi_m^2+\phi_n^2}{2}\right)\right\}$$

$$= -\frac{\phi_s}{a^2}\mu''(\phi_{sss}+\phi_{mms}+\phi_{nns}) - \frac{\mu''}{a^2}(\phi_{sst}+\phi_{mmt}+\phi_{nnt})$$

$$+ \frac{\gamma-1}{a^2}\mu''(\phi_{ss}^2+\phi_{mm}^2+\phi_{nn}^2+2\phi_{ss}\phi_{mm}+2\phi_{ss}\phi_{nn} +2\phi_{mm}\phi_{nn}) + \frac{\gamma-1}{a^2}4\mu(\phi_{sm}^2+\phi_{sn}^2+\phi_{mn}^2-\phi_{ss}\phi_{mm}$$

$$- \phi_{ss}\phi_{nn} - \phi_{mm}\phi_{nn}) + \frac{\gamma-1}{a^2}\left\{\left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial m}\hat{e}_m + \frac{\partial}{\partial n}\hat{e}_n\right)\right.$$

$$\left.\cdot\left[k(T)_s\hat{e}_s + k(T)_m\hat{e}_m + k(T)_n\hat{e}_n\right]\right\} \qquad (33)$$

Here, $\hat{e}_s$, $\hat{e}_m$, and $\hat{e}_n$ are mutually orthogonal unit vectors in s, m and n direction respectively. Since all quantities are normalized with their critical thermodynamic values, the isentropic speed of sound can be expressed as

$$T = a^2 = \frac{a^2}{a_*^2} = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}[\phi_s^2 + \phi_m^2 + \phi_n^2] \qquad (34)$$

Then

$$(a^2)_s = -(\gamma-1)(\phi_s\phi_{ss} + \phi_m\phi_{ms} + \phi_n\phi_{ns}) \qquad (35)$$

$$(a^2)_m = -(\gamma-1)(\phi_s\phi_{sm} + \phi_m\phi_{mm} + \phi_n\phi_{nm}) \qquad (36)$$

$$(a^2)_n = -(\gamma-1)(\phi_s\phi_{sn} + \phi_m\phi_{mn} + \phi_n\phi_{nn}) \qquad (37)$$

Hence,

$$(a^2)_{ss} = -(\gamma-1)(\phi_{ss}^2 + \phi_s\phi_{sss} + \phi_{ms}^2 + \phi_m\phi_{mss} + \phi_{ns}^2 + \phi_n\phi_{nss}) \qquad (38)$$

$$(a^2)_{mm} = -(\gamma-1)(\phi_{sm}^2 + \phi_s\phi_{smm} + \phi_{mm}^2 + \phi_m\phi_{mmm} + \phi_{nm}^2 + \phi_n\phi_{nmm}) \qquad (39)$$

$$(a^2)_{nn} = -(\gamma-1)(\phi_{sn}^2 + \phi_s\phi_{snn} + \phi_{mn}^2 + \phi_m\phi_{mnn} + \phi_{nn}^2 + \phi_n\phi_{nnn}) \qquad (40)$$

Since $\phi_m = \phi_n = 0$; then the nondimensional mass conservation (Eq. 33) can be written as $\mu = \mu/\mu_\infty$; $\mu'' = \mu''/\mu_\infty$ and $k = k/k_\infty$

$$\rho\left\{(1 - M^2)\phi_{ss} + \phi_{mm} + \phi_{nn}] - \frac{1}{a^2}(\phi_{tt} + 2\phi_s\phi_{st})\right\}$$

$$= \frac{1}{Re}\left\{-\frac{\phi_s}{a^2}\left[\mu'' + (\gamma-1)\frac{k}{C_p}\right](\phi_{sss} + \phi_{smm} + \phi_{snn})\right.$$

$$+ \frac{\gamma-1}{a^2}\left(\mu'' - \frac{k}{C_p}\right)(\phi_{ss}^2 + \phi_{mm}^2 + \phi_{nn}^2)$$

$$+ 2\frac{\gamma-1}{a^2}(\mu'' - 2\mu)(\phi_{ss}\phi_{mm} + \phi_{ss}\phi_{nn} + \phi_{mm}\phi_{nn})$$

$$- 2\frac{\gamma-1}{a^2}\left(\frac{k}{C_p} - 2\mu\right)(\phi_{sm}^2 + \phi_{sn}^2 + \phi_{mn}^2)$$

$$\left.\cancel{-\frac{1}{a^2}(\phi_{tt} + 2\phi_s\phi_{st})} - \frac{\mu''}{a^2}(\phi_{sst} + \phi_{mmt} + \phi_{nnt})\right\} \qquad (41)$$

Since $\frac{(\gamma-1)}{\gamma R} = \frac{1}{C_p}$, it is convenient to define a longitudinal [3] Prandtl number P" as $\frac{1}{P''} = \frac{k}{C_p\mu''}$ and the Reynolds number Re as $Re = \rho_* a_* L/\mu_\infty$.

Hence, the formula for mass conservation in an unsteady, irrotational flow of heat conducting, calorically perfect, viscous gas without body forces, radiation heat transfer, and mass sources or sinks is:

$$\rho\left\{\left[(1-M^2)\phi_{ss} + \phi_{mm} + \phi_{nn}\right] - \frac{1}{a^2}\left(\phi_{tt} + 2\phi_s\phi_{st}\right)\right\}$$

$$= \frac{1}{Re}\Big\{ -\frac{\phi_s}{a^2}\mu''\left(1+\frac{\gamma-1}{P''}\right)(\phi_{sss} + \phi_{smm} + \phi_{snn})$$

$$+ \frac{\gamma-1}{a^2}\mu''\left(1 - \frac{1}{P''}\right)(\phi_{ss}^{\,2}+\phi_{mm}^{\,2}+\phi_{nn}^{\,2})$$

$$+ 2\frac{\gamma-1}{a^2}\mu''\left(1-\frac{2\mu}{\mu''}\right)(\phi_{ss}\phi_{mm} + \phi_{ss}\phi_{nn} + \phi_{mm}\phi_{nn})$$

$$- 2\frac{\gamma-1}{a^2}\mu''\left(\frac{1}{P''} - 2\frac{\mu}{\mu''}\right)(\phi_{sm}^{\,2} + \phi_{sn}^{\,2} + \phi_{mn}^{\,2})$$

$$- \frac{\mu''}{a^2}(\phi_{sst} + \phi_{mmt} + \phi_{nnt})\Big\} \qquad (42)$$

We will refer to Eq. 42 as the Physically Dissipative Potential (PDP) equation. If the flow is steady and two-dimensional, Eq. 42 reduces to

$$\rho[(1-M^2)\phi_{ss} + \phi_{nn}] = -\frac{\phi_s}{a^2}\frac{\mu''}{Re}\left(1+\frac{\gamma-1}{P''}\right)(\phi_{sss}+\phi_{snn})$$

$$+ \frac{\gamma-1}{a^2}\frac{\mu''}{Re}\left(1 - \frac{1}{P''}\right)(\phi_{ss}^{\,2}+\phi_{nn}^{\,2})$$

$$+ 2\frac{\gamma-1}{a^2}\frac{\mu''}{Re}\left(1 - 2\frac{\mu}{\mu''}\right)\phi_{ss}\phi_{nn} - 2\frac{\gamma-1}{a^2}\frac{\mu''}{Re}\left(\frac{1}{P''} - 2\frac{\mu}{\mu''}\right)\phi_{sn}^{\,2} \qquad (43)$$

When the viscosity is negligible, the entire right hand side becomes zero and Eq. 43 converts to an FPE.
This highly nonlinear expression can now be compared with the classical V-T small perturbation equation [3]

$$-(\gamma+1)\,\psi_x\,\psi_{xx} + \psi_{yy} = \frac{-1}{Re}\left(1 + \frac{\gamma-1}{P''}\right)\psi_{xxx} \qquad (44)$$

or the more complete pseudo-transonic [9,10] V-T equation

$$[1-M_\infty^2-(\gamma+1)M_\infty^4\psi_x]\psi_{xx} + \psi_{yy} = \delta\psi_{xxx} \qquad (45)$$

Here $\delta>0$ is a small diffusion coefficient and ψ is the perturbation velocity potential: $|\psi_x| \ll 1$.
Again, note that equations 42 and 43 satisfy energy, momentum, and mass conservation and that they were derived without the assumptions of small perturbations and the consequent linearizations.
It is obvious that PDP equation is considerably more complex than the V-T equation, since PDP equation retains all the nonlinearities. Nevertheless, the linear dissipation terms in both equations are practically the same.

If Eq. 24 is reduced to a one-dimensional steady version, the result is

$$T\nabla s - \nabla h_o = \frac{-\mu''\,\phi_{xxx}}{\rho} \qquad (46)$$

Next, the conservation of energy (Eq. 6) can be written as

$$(\vec{V}\cdot\nabla)h = -\frac{\partial h}{\partial t} + \frac{1}{\rho}\left\{\frac{\partial p}{\partial t} + (\vec{V}\cdot\nabla)p + \Phi + \nabla\cdot k\nabla T\right\} \qquad (47)$$

The conservation of momentum (Eq. 13) can be expressed as

$$\nabla p = -\rho\frac{\partial \vec{V}}{\partial t} - \rho(\vec{V}\cdot\nabla)\,\vec{V} + \left\{2\nabla(\mu\nabla\cdot\vec{V}) - \nabla\times(\mu\nabla\times\vec{V}) + \nabla(\lambda(\nabla\cdot\vec{V}))\right\} \qquad (48)$$

if the body forces are negligible. Substituting Eq. 48 into Eq. 47, using the vector identity (Eq. 14) and keeping only the steady terms, yields

$$\rho\vec{V}\cdot\nabla h_o = \vec{V}\cdot\left\{2\nabla(\mu\nabla\cdot\vec{V}) - \nabla\times(\mu\nabla\times\vec{V}) + \nabla(\lambda(\nabla\cdot\vec{V}))\right\}$$

$$+ \Phi + \nabla\cdot k\nabla T + \rho\vec{V}\cdot(\vec{V}\times(\nabla\times\vec{V})) \qquad (49)$$

For one-dimensional flow (which is always irrotational), Eq. 49 reduces to

$$\frac{\partial h_o}{\partial x} = \frac{\mu''}{\rho}\left(1 - \frac{1}{P''}\right)\left(\frac{\phi_{xx}^{\,2}}{\phi_x} + \phi_{xxx}\right)\frac{(\gamma-1)}{Re} \qquad (50)$$

Eq. 50 shows that for steady one-dimensional flows the stagnation enthalpy is constant through a shock wave only when $P'' = 1$ is satisfied. Since $P'' = P\mu''/\mu$ and $P = 3/4$ for a diatomic gas, it follows that this is true only when Stokes hypothesis $\mu''/\mu = 4/3$ is used.

NUMERICAL EXAMPLES

With Eq. 50 and Eq. 46 the entropy variation through a normal shock can be found. Equation 50 was integrated assuming Stokes hypothesis (Figure 1). The final entropy jump across the shock wave satisfies the Rankine-Hugoniot jump [11] condition for entropy. Nevertheless, the entropy exhibits a sharp spike in the middle of the shock (Figure 1). From the entropy generation equation it is easy to explain this phenomena. The viscous dissiptation Φ is always positive. The heat flux ($k\nabla^2 T$) is positive only until the middle of the shock; downstream from the middle of the shock it becomes negative thus lowering the entropy.

For the purpose of testing the accuracy and evaluating the sensitivity of the PDP equation (Eq. 42), its one dimensional steady version was used

$$\rho(1-M^2)\phi_{ss} = \frac{-\mu''}{\mathcal{R}e}\left(1 + \frac{\gamma-1}{P''}\right)\frac{\phi_s}{a^2}\,\phi_{sss}$$

$$+ \frac{\mu''}{\mathcal{R}e}\frac{\gamma-1}{a^2}\left(1 - \frac{1}{P''}\right)(\phi_{ss})^2 \qquad (51)$$

Since

$$\rho(1-M^2) = \frac{\rho}{a^2}\,(a^2 - \phi_s^{\,2}) \qquad (52)$$

and the local speed of sound, a, is defined in Eq. 34, it follows that Eq. 51 after multiplication with a^2 can be rewritten as

$$\frac{-\mu''}{\mathcal{R}e}\left(1+\frac{\gamma-1}{P''}\right)\phi_x\phi_{xxx} + \frac{\mu''}{\mathcal{R}e}(\gamma-1)\left(1 - \frac{1}{P''}\right)\phi_{xx}^{\,2}$$

$$- \rho\frac{\gamma+1}{2}\,(1-\phi_x^{\,2})\phi_{xx} = 0 \qquad (53)$$

Equation 53 was numerically integrated using a fourth order accurate Runge-Kutta scheme. The integration interval was $0 \le x \le 1$ and the step size was $\Delta x = .000001$. The values for P and $\mu''/\mathcal{R}e$ were P = 3/4 and $\mu''/\mathcal{R}e = 0.0000195$. Values of the physical properties μ, k, γ and P are well documented in existing literature, but experimentally obtained values for λ and P" differ by orders of magnitude. For example, Stokes hypothesis states that the bulk viscosity is zero ($\mu_B = \frac{2}{3}\mu + \lambda = 0$).

Hence, $\lambda = -\frac{2}{3}\mu$ is the most frequently used value for the secondary viscosity. Nevertheless, from data compiled by Truesdell [12], $\mu_B \approx \frac{2}{3}\mu$ for air suggesting that $\lambda = 0$.

To investigate the effect of different values of λ on the solution of Eq. 53, several computer runs were performed. When Eq. 53 is solved using Stokes hypothesis ($\lambda = -\frac{2}{3}\mu$), then the results will match Rankine-Hugoniot shock jumps (Figure 2). In order to illustrate the influence of secondary viscosity, λ, on the magnitude of the shock jump, a number of numerical tests were performed with various values of λ/μ and a fixed value of upstream critical Mach number, $(\phi_s)_1 = 1.2$. The results of this comparison (Figure 3) confirm the intuitive expectation that smaller values of λ cause steepening of the shock wave. Variation of the critical Mach number $(\phi_s)_2$ downstream of the normal shock as a function of the secondary viscosity μ'' is shown in Figure 4. Notice that Rankine-Hugoniot jump conditions will be obtained when $\lambda/\mu = -2/3$ and that the isentropic shock jump conditions [11] will be obtained when $\lambda/\mu \cong -2$. Thus, PDP accepts Rankine-Hugoniot and isentropic shocks as a part of its general solution.

CONCLUSIONS

A new analytic model was derived that combines mass, momentum and energy conservation in a single Physically Dissipative Potential equation for nonsteady, irrotational flow of viscous, heat conducting, calorically perfect gases without body forces. The governing equation is a third order highly nonlinear partial differential equation which accurately predicts strengths and structures of the shock waves. This equation can be used instead of the Full Potential Equation as a more appropriate model for transonic shocked flow computations and especially for the more appropriate modelling and analysis of numerical dissipation. In addition, it can be used in nonlinear acoustics where it is important to accurately predict the attenuation of sound waves.

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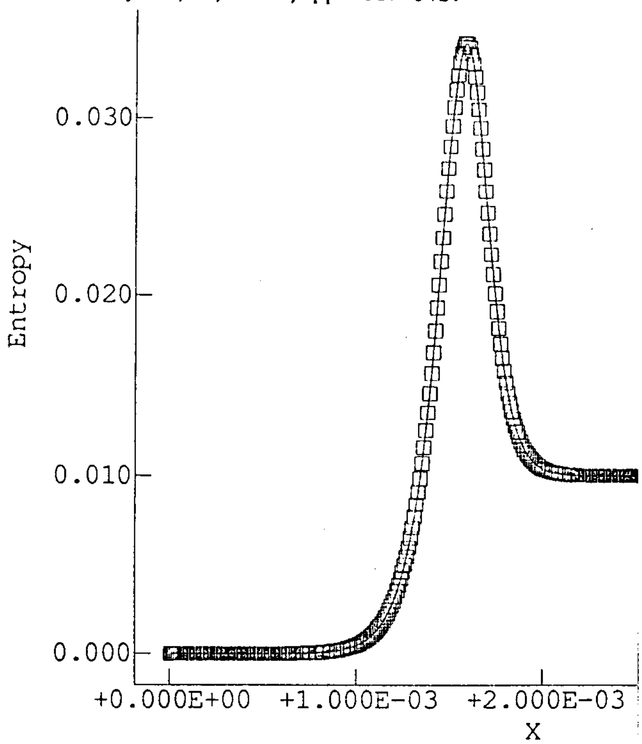

Figure 1. Entropy variation through a one-dimensional shock produced by the PDP equation with $M_{*1}=1.2$; $P=3/4$; $\gamma=1.4$; $\mu/Re=0.00001985$; $\lambda/\mu=-2/3$.

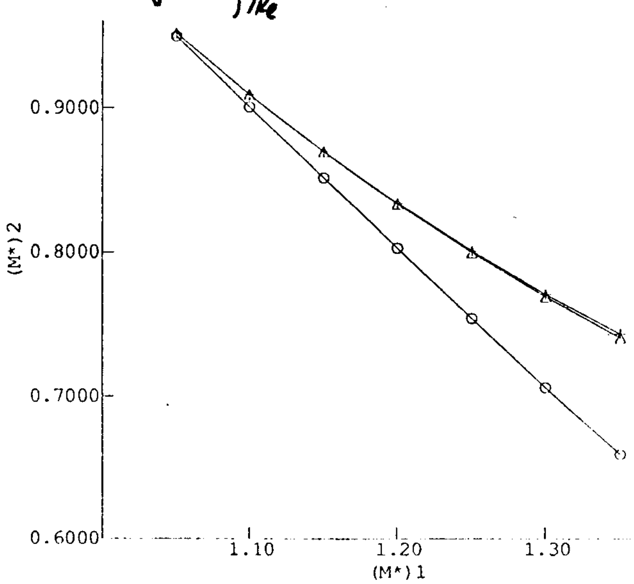

Figure 2. One-dimensional shock jumps for full potential equation (o-o-o-o), PDP equation with $\lambda/\mu=-2/3$ (△-△-△) and for Rankine-Hugoniot shocks(+-+-+)

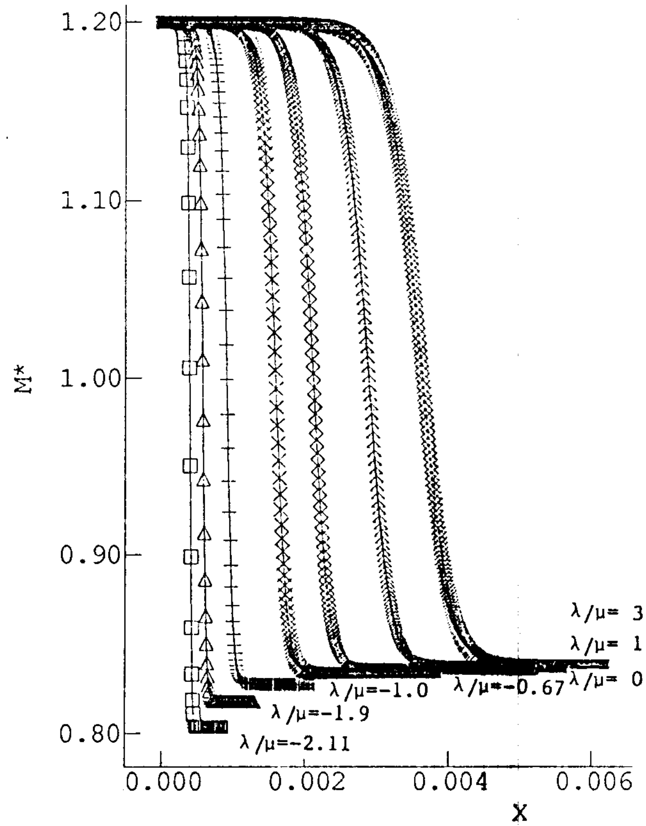

Figure 3. Shock profiles computed with different values for λ/μ. Notice varying shock strengths and thickness.

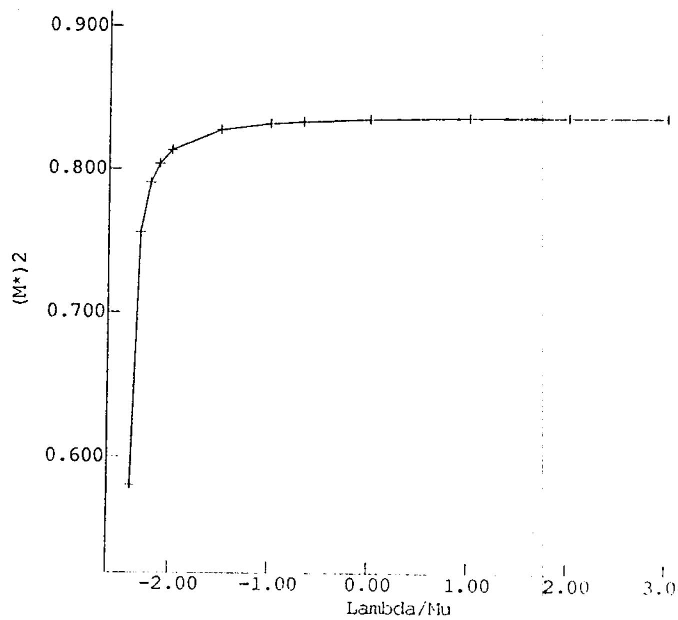

Figure 4. Computed values of M_* behind a shock for various values of λ/μ and $M_{*1}=1.2$. Stokes hypothesis ($\lambda/\mu=-2/3$) produces Rankine-Hugoniot shock jumps.