ARTIFICIAL MASS CONCEPT AND TRANSONIC VISCOUS FLOW EQUATION

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ABSTRACT. By varying the grid clustering on the surface of an airfoil, it was observed that symmetric shocked solutions develop with a nonunique shock strength and location when numerically solving the full potential equation.

It is shown analytically that the conventional form of artificial density (or viscosity) produces a number of truly nonlinear terms which are suspected to be the cause of the nonuniqueness for all the finite grid sizes. A concept of artificial mass flow is shown to be suitable for analytically evaluating a new exact form of the switching function that eliminates all the nonlinear terms for any value of the local Mach number. The resulting expanded full potential equation then becomes a third order partial differential equation of permanently parabolic type resembling Sichel's transonic viscous flow equation. Consequently, our expanded full potential equation does not require the introduction of the customarily used artificial time concept.

I. INTRODUCTION. The numerical techniques for solving full potential equation modelling transonic flows are presently based on two similar concepts: an explicitly added artificial viscosity [1] and an implicitly modified artificial density [2]. Both techniques should create additional terms in the full potential equation in such a way that they nullify the numerical error introduced when using upstream rotated finite differencing in the locally supersonic regions.

If these artificial dissipative terms are introduced in a divergence free form, it was believed that the numerical solution will be unique [3]. Nevertheless, in recent years it was observed that the question of uniqueness is not resolved when isentropic discontinuities (shocks) are present in the solution. It has been shown that the finite difference formulation [4] of the artificial viscosity can somewhat affect the location and the strength of the isentropic discontinuities without causing numerical instability problems. The artificial damping was monotonically introduced in conservative form across the sonic line at all subsonic parts of the flow domain where the local value of the Mach number exceeded an arbitrary prescribed cut-off value [5]. A disturbing conclusion was that the numerical solution depends upon the choice of the initial guess for the potential field, varied choices for which causing the numerical solution to converge to strikingly different answers [5] if shocks are present in the field. At the same time it has been demonstrated that the nonunique solutions are not dependent on the type of the grid used nor are they dependent on the grid resolution for the grid sizes commonly used. The conclusion was that these were the correct nonunique solutions of the exact full potential equation.

We numerically experimented with a number of two-dimensional full potential cascade codes [6,7,8] and consistently observed non-unique symmetric shocked solutions. The test case was a nonstaggered cascade of NACA0012 airfoils spaced at 3.6 chord lengths apart. The free stream Mach number was $M = 0.8$, and the free stream angle of attack was zero. First order artificial viscosity in a fully conservative form [3,9] was used in each of the codes. Numerical solutions were obtained on a sequence of four successively refined O-type and C-type non-orthogonal boundary fitting geometrically periodic grids. When the grid points on the airfoils surface were equidistantly spaced, the iterative procedure converged to a symmetric solution with the shock located at approximately 80 percent of the chord.

Then the same sequence of grids with the same number of grid points was used with the only exception that the grid points on the airfoil's surface were symmetrically clustered closer to the leading edge and the trailing edge. The result was a symmetric solution with the shocks located at approximately 72 percent of the chord. It should be noted that the boundary and the periodicity conditions were enforced explicitly and exactly and that the airfoil surface Mach number drop across the shocks was in both test cases in close agreement with the exact one-dimensional isentropic shock conditions. These results were obtained with both O-type and C-type grids while using the same (symmetric) initial guess for the potential field, the same number of iterations and the same relaxation factors on each of the grids.

II. ANALYSIS OF ARTIFICIAL DENSITY CONCEPT. The following derivations should suggest the probable cause for nonuniqueness of the shocked solution of the discretized form of the full potential equation.

Mass conservation equation

$$\vec{\nabla}\cdot(\rho\vec{\nabla}\phi) = \left(\frac{\partial}{\partial s}\hat{e}_s + \frac{\partial}{\partial n}\hat{e}_n\right)\cdot(\rho\phi_{,s}\hat{e}_s + \rho\phi_{,n}\hat{e}_n) = 0 \qquad (1)$$

where comma denotes partial differentiation, can be written in its canonical form [10]

$$\vec{\nabla}\cdot(\rho\vec{\nabla}\phi) = \rho[(1-M^2)\phi_{,ss} + \phi_{,nn}] = 0 \tag{2}$$

If all the flow variables are nondimensionalized with their respective critical properties then

$$\frac{\rho_{,s}}{\rho} = -\frac{M^2}{M_*} M_{*,s} \tag{3}$$

If an artificial density of the general form

$$\tilde{\rho} = \rho - \Delta s\ \mu\rho_{,s} \tag{4}$$

is introduced, it can be shown that

$$\frac{\tilde{\rho}_{,s}}{\tilde{\rho}} = \frac{\rho_{,s}}{\rho} + \frac{1}{1 + \Delta s\ \mu \frac{M^2}{M_*} M_{*,s}} [\Delta s\ \mu \frac{M^2}{M_*} M_{*,s} + \Delta s(\mu \frac{M^2}{M_*})_{,s} M_{*,s}] \tag{5}$$

Hence, the mass conservation equation becomes

$$\vec{\nabla}\cdot(\tilde{\rho}\vec{\nabla}\phi) = \tilde{\rho}\{[1-M^2)\phi_{,ss} + \phi_{,nn}] + \Delta s\ \mu M^2\ \phi_{,sss} + E\} = 0 \tag{6}$$

where

$$E = \frac{\Delta s\ M_{*,s}}{1 + \Delta s\ \mu \frac{M^2}{M_*} \phi_{,ss}} [(\mu \frac{M^2}{M_*})_{,s} - (\mu \frac{M^2}{M_*})^2\ \Delta s\ M_{*,ss}] \tag{7}$$

Then the undesirable term in the expanded full potential equation is

$$E = \frac{\Delta s\ M^2}{1 + \Delta s\ \mu \frac{M^2}{M_*} \phi_{,ss}} [-\Delta s\ \mu^2 \frac{M^2}{M_*} \phi_{,ss}\ \phi_{,sss}$$

$$+ \mu((1+\gamma)\ \frac{M^2}{M_*^2} - 1)\ \frac{(\phi_{,ss})^2}{M_*} + \mu_{,s}\ \phi_{,ss}] \tag{8}$$

The switching function μ is customarily [1,2] assigned the value

$$\mu = \mu_{JH} = 1 - \frac{1}{M^2} \tag{9}$$

which is obtained from the condition that the term

$$\Delta s\ \mu\ M^2\ \phi_{,sss} \tag{10}$$

should be approximately the same magnitude as the terms introduced by the upstream differencing of $\phi_{,ss}$ in the locally supersonic regions of the flow field. If μ_{JH} is used in E, the result is a cluster of truly nonlinear terms

$$E = \frac{\Delta s\ M^2}{1 + \Delta s(M^2-1)\ \frac{\phi_{,ss}}{\phi_{,s}}} [-\Delta s(M^2-1)(1- \frac{1}{M^2})\ \frac{\phi_{,ss}\ \phi_{,sss}}{\phi_{,s}}$$

$$+ (1- \frac{1}{M^2})((1+\gamma)\ \frac{M^2}{(\phi_{,s})^2} - 1)\ \frac{(\phi_{,ss})^2}{\phi_{,s}} + (1+\gamma)\ \frac{(\phi_{,ss})^2}{(\phi_{,s})^3}]$$

(11)

It is obvious that the conventional value of μ fails to make $E = 0$ for any value of local flow Mach number equal to or greater than one. Actually, when using μ_{JH} one ends up solving a nonlinear partial differential equation even on the sonic line where eq. 6 reduces to

$$\phi_{,nn} + \Delta s(1+\gamma)(\phi_{,ss})^2 = 0 \tag{12}$$

The substantiated explanation of the influence of the nonlinear terms on the final solution of the expanded full potential equation is not available at the present time. Nevertheless, it could be speculated that these nonlinear terms have some properties of solitons thus causing the numerically observed non-unique solutions for all realistic non-zero grid sizes. The influence of the nonlinear terms can be certainly affected by the particular finite differencing applied in the evaluation of the derivatives constituting the artificial viscosity [4] or artificial density [11]. Different shocked solutions can also be achieved by varying the expression for the switching function μ [12,13].

Full potential equation does not involve any dissipative mechanism on the basis of which expansion discontinuities could be eliminated. The numerically created artificial viscosity [1] and density [2] terms coupled with an artificial time concept [1] represented the basis of almost all transonic potential flow computations performed over the past decade.

By allowing for an insignificant vorticity generation in a limiting process applied to a small disturbance transonic potential equation, Sichel [14] has derived a "viscous transonic equation."

$$K_v \phi_{,xxx} + (K_\infty - \phi_{,x})\phi_{,xx} + \phi_{,yy} = 0 \tag{13}$$

where

$$K_v = (1 + \frac{\gamma-1}{Pr})/[\tau(1+\gamma)M_\infty^2]^{2/3}Re \tag{14}$$

and

$$K_\infty = (1 - M_\infty^2)/[\tau(1+\gamma)M_\infty^2]^{2/3} \tag{15}$$

Here, M_∞ is the free stream Mach number, γ is the ratio of specific heats, Pr and Re are Prandtl and Reynolds numbers, respectively, and τ is half the airfoil thickness ratio.

Since $K_v > 0$ this equation is parabolic and Chin [15,16,17] solved it numerically without any need to introduce the artificial time. Although it is better suited for modelling the physical details at the shocks, eq. 13 cannot be recast in a divergence free form.

III. ARTIFICIAL MASS FLUX CONCEPT. It is possible, nevertheless, to derive an expanded full potential equation that will always be of a parabolic type and will be readily expressible in a divergence free form. The idea is to expand the mass flux (rather than density or the speed of sound [2] alone) in a Taylor series. Such a modified mass conservation equation

$$\vec{\nabla}\cdot\overline{(\rho\vec{\nabla}\phi)} = 0 \tag{16}$$

can be expressed in the locally streamline aligned orthogonal coordinate system (s,n) as

$$\left(\frac{\partial}{\partial_s}\hat{e}_s + \frac{\partial}{\partial_n}\hat{e}_n\right)\cdot[(\rho\phi_{,s} - \Delta s\,\mu(\rho\phi_{,s})_{,s})\hat{e}_s + (\rho\phi_{,n})\hat{e}_n] = 0 \tag{17}$$

Hence,

$$\rho[(1-M^2)\phi_{,ss} + \phi_{,nn}] - \Delta s\,\rho[\mu_{,s}(\phi_{,ss} + \frac{\rho_{,s}}{\rho}\phi_{,s})$$
$$+ \mu(\frac{\rho_{,ss}}{\rho}\phi_{,s} + 2\frac{\rho_{,s}}{\rho}\phi_{,ss} + \phi_{,sss})] = 0 \tag{18}$$

Because

$$\rho = [\frac{\gamma+1}{2} - \frac{\gamma-1}{2}(\phi_{,s})^2]^{1/(\gamma-1)}$$
$$\tag{19}$$
$$\phi_{,s} = M_*$$

It follows that the artificial mass produces

$$\rho\{[(1-M^2)\phi_{,ss} + \phi_{,nn}] + \Delta s\ \mu(M^2-1)\phi_{,sss}\}$$

$$- \Delta s\ \rho\{(1-M^2)\mu_{,s}\phi_{,ss} - \mu[(1+\gamma)\ \frac{M^4}{M_*^3} + \frac{M^2}{M_*} - \frac{M^4}{M_*}](\phi_{,ss})^2\} = 0 \tag{20}$$

The second brace contains undesirable nonlinear terms. The value of the switching function μ is determined in such a way as to eliminate them entirely. Hence,

$$\frac{du}{\mu} = \frac{-2M_*}{(M_*^2-1)}\ \frac{d\ M_*}{(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\ M_*^2)} - (\frac{2}{\gamma+1})\ \frac{M_*\ dM_*}{(M_*^2-1)}$$

$$+ (\frac{2}{\gamma+1})\ \frac{M_*^3}{(M_*^2-1)}\ \frac{d\ M_*}{(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\ M_*^2)} \tag{21}$$

The result is

$$\mu = \frac{1}{(M_*^2-1)}\ (a^2)^{\frac{2-\gamma}{\gamma-1}} \tag{22}$$

Because of the relation

$$\frac{M^2-1}{M_*^2-1} = (\frac{\gamma+1}{2})(a^2)^{-1} \tag{23}$$

it follows that the modified mass conservation becomes

$$\rho\{[(1-M^2)\phi_{,ss} + \phi_{,nn}] + \Delta s(\frac{\gamma+1}{2})(\frac{1}{\rho})\phi_{,sss}\} = 0 \tag{24}$$

This expanded full potential equation is of parabolic type for any value of Mach number and its variable diffusion coefficient does not vanish for any finite value of the Mach number. Note the striking similarity between eq. 24 and eq. 13 and the fact that eq. 24 can be integrated without a need for artificial time variable [1]. The divergence free form of eq. 24 can be achieved as follows.
Let

$$A = \rho\phi_{,s} - \Delta s\ \mu(\rho\phi_{,s})_{,s} \tag{25}$$

Hence,

$$A = \phi_{,s}[\rho - \Delta s\ \mu(1 - \frac{1}{M^2})\rho_{,s}] \tag{26}$$

and the modified mass conservation (eq. 16) can be expressed in its fully conservative form as

$$\vec{\nabla}\cdot(\tilde{\rho}\vec{\nabla}\phi) = (\tilde{\rho}\phi_{,x})_{,x} + (\tilde{\rho}\phi_{,y})_{,y} = 0 \tag{27}$$

where the exact form of the artificial density is

$$\tilde{\rho} = \rho - \Delta s\ \mu(\frac{M^2-1}{M^2})\rho_{,s} \tag{28}$$

or

$$\tilde{\rho} = \rho - \Delta s(\frac{\gamma+1}{2})\ \frac{1}{M^2}\ (\frac{1}{\rho})\rho_{,s} \tag{29}$$

IV. SUMMARY. It has been analytically proven that the usual formulation of artificial density and viscosity terms leads to an introduction of truly nonlinear terms whose effects are suspected of causing certain numerical errors and inconsistencies in the numerical computation of transonic potential flows. A new concept of artificial mass flux was shown to produce only linear artificial dissipation that has the same basic character as a governing equation for physically viscous transonic flow. The artificial mass can be easily reformulated in terms of a new artificial density in a fully conservative form.

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