Magnetohydrodynamic simulations using radial basis functions

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A B S T R A C T
To overcome the computational mesh quality difficulties, mesh-free methods have been developed. One
of the most popular mesh-free kernel approximation techniques is radial basis functions (RBFs). Initially,
RBFs were developed for multivariate data and function interpolation. It is well-known that a good inter-
polation scheme also has great potential for solving partial differential equations. In the present study,
the RBFs are used to interpolate stream-function and temperature in a two-dimensional thermal buoy-
ancy flow acted upon by an externally applied steady magnetic field. Use of mesh-free methods promises
to significantly reduce the computing time, especially for the complex classes of problems such as
magnetohydrodynamics.

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1. Introduction

Radial basis functions are essential ingredients of the tech-
niques generally known as “meshless methods”. In one way or an-
other all meshless techniques require some sort of radial function
to measure the influence of a given location on another part of the
domain. The use of radial basis functions (RBF) followed by collo-
cation, a technique first proposed by Kansa [1], after the work of
Hardy [2] on multivariate approximation, is now becoming an
established approach and various applications to problems of
structures and fluids have been made in recent years. See, for
example, Leitão [3,4].

Kansa’s method (or asymmetric collocation) starts by building
an approximation to the field of interest (normally displacement
components) from the superposition of radial basis functions
(globally or compactly supported) conveniently placed at points
in the domain (and, or, at the boundary).

The unknowns (which are the coefficients of each RBF) are ob-
tained from the (approximate) enforcement of the boundary condi-
tions as well as the governing equations by means of collocation.
Usually, this approximation only considers regular radial basis
functions, such as the globally supported multiquadrics or the

Radial basis functions (RBFs) may be classified into two main
groups:

1. the globally supported ones namely the multiquadric (MQ,
\[ \sqrt{(x-x_j)^2 + c_j^2} \], where \( c_j \) is a shape parameter), the inverse mul-
quadric, thin plate splines, Gaussians, etc;
2. the compactly supported ones such as the Wendland [5] family
(for example, \( (1-\frac{r}{\alpha})^6 + p(r) \) where \( p(r) \) is a polynomial and
\( (1-r)^n \) is 0 for \( r \) greater than the support).

In a very brief manner, interpolation with RBFs may take the
form:
\[ s(x_i) = f(x_i) = \sum_{j=1}^{N} \alpha_j \phi(|x_i-x_j|) + \sum_{k=1}^{N} \beta_k p_k(x_i) \]  \hspace{1cm} (1)
where \( f(x_i) \) is known for a series of points \( x_i \) and \( p_k(x_i) \) is one of the \( \hat{N} \)
terms of a given basis of polynomials [6]. This approximation is
solved for the \( \alpha_j \) unknowns from the system of \( N \) linear equations,
subject to the conditions (for the sake of uniqueness)
\[ \sum_{j=1}^{N} \alpha_j p_k(x_j) = 0. \]  \hspace{1cm} (2)

By using the same reasoning, it is possible to extend the interpola-
tion concept to that of finding the approximate solution of partial
differential equations. This is made by applying the corresponding
differential operators to the RBFs and then to use collocation at an
appropriate set of boundary and domain points. In short, the non-
symmetrical collocation is the application of the domain and
boundary differential operators \( L_I \) and \( L_B \), respectively, to a set of
\( N-M \) domain collocation points and \( M \) boundary collocation points.
From this, a system of linear equations of the following type may be
obtained:
\[ X_{j,j+1} \]
Nomenclature

\( \mathbf{B} \) magnetic field
\( B_0 \) externally applied magnetic field of reference
\( c \) shape parameter used in the radial basis functions
\( f \) exact value of the functions at the interpolation points
\( g \) gravity vector
\( Gr \) Grashof number
\( Ha \) Hartmann number
\( J \) electric current density
\( L \) length of the cavity
\( M, N \) number of centers in the \( x \) and \( y \) direction, used in the RBF approximation
\( P \) pressure
\( Pr \) Prandtl number
\( r \) euclidian norm between any two points
\( \text{RBF} \) radial basis function
\( s \) interpolated functions
\( T \) temperature
\( T_0 \) hot temperature
\( T_c \) cold temperature
\( T_\infty \) reference temperature

\[ L \mu_0 (x) = \sum_{j=1}^{N} \alpha_j L \phi(x_j - c_j) + \sum_{k=1}^{N} \beta_k L p_b(x_k) \]

\[ L B u_b(x) = \sum_{j=1}^{N} \alpha_j L B \phi(x_j - c_j) + \sum_{k=1}^{N} \beta_k L B p_b(x_k) \]

subject to the conditions \( \sum_{i=1}^{N} \alpha_i p_b(x_i) = 0 \) where the \( \alpha_j \) and \( \beta_k \) unknowns are determined from the satisfaction of the domain and boundary constraints at the collocation points.

2. Magnetohydrodynamic equations

In this paper we consider the laminar, steady and incompressible fluid flow of an electrically conducting fluid within a square cavity whose top and bottom walls are kept insulated and left and right walls are subjected to different and constant temperatures. The fluid properties are considered constants and the difference of temperature will originate a buoyancy force, which is modeled using the Boussinesq’s approximation. The fluid is permeated by a constant magnetic field which will create an additional buoyancy force. The governing equations are the conservation of mass, momentum, energy and conservation of electric charges, Ohm’s law and Ampere–Maxwell’s law [7,8] in a moving medium

\[ \nabla \cdot \mathbf{V} = 0 \quad \text{(4.a)} \]

\[ (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \frac{J \times \mathbf{B}}{\rho} + \nu \nabla^2 \mathbf{V} - \mu_0 \mathbf{g} (T - T_0) \quad \text{(4.b)} \]

\[ (\mathbf{V} \cdot \nabla) T = \alpha_t \nabla^2 T + \frac{J}{\rho \kappa} \cdot (-\nabla \phi + \mathbf{V} \times \mathbf{B}) \quad \text{(4.c)} \]

\[ \nabla \cdot \mathbf{J} = 0 \quad \text{(4.d)} \]

\[ \mathbf{J} = \sigma (-\nabla \phi + \mathbf{V} \times \mathbf{B}) \quad \text{(4.e)} \]

\[ \nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} \quad \text{(4.f)} \]

where \( \mu_0 \) is the magnetic permeability of the vacuum. Note that the Lorentz force is represented in the momentum equations through the vector product of the electric current density and the magnetic field. Also, in the above equations, the effects of polarization and magnetization were neglected [7].

As discussed by Garandet et al. [9], the harmonic equation for the electric potential, \( \nabla \phi = 0 \), is satisfied in the fluid as in the neighboring solid media. The unique solution of the harmonic equation is \( \nabla \phi = 0 \) since there is always an electrically insulating boundary on which \( \partial \phi / \partial n = 0 \) around the enclosure. It follows that the electric field vanishes everywhere. Also, it is easy to show by substituting (4.f) into (4.d), that for a two-dimensional flow, the conservation of the electric charges is automatically satisfied, reducing the final set of equations to (4.a–c, e), with \( \nabla \phi = 0 \).

In the present paper, the magnetic Reynolds number \( Rm = \mu_0 \sigma L H_0 / \nu \) is very small. Also, the effects of Joule heating and viscous dissipation are supposed to be very small, so we can neglect the second term on the right hand side of (4.c).

Defining the stream-function

\[ u = \frac{\partial \psi}{\partial y} \quad v = \frac{\partial \psi}{\partial x} \quad \text{(5.a, b)} \]

and the following dimensionless quantities:

\[ x' = x/L \quad y' = y/L \quad \psi' = k/T (T - T_0)/(T_b - T_c) \quad \text{(6.a-d)} \]

where \( L \) is the length of the cavity and \( T_0 \) and \( T_b \) are the cold and hot temperatures of the container walls, respectively, we obtain, combining (4.a) and (4.b)

\[ \frac{\partial \psi'}{\partial x'} \left( \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial y'^2} \right) = \frac{\partial \psi'}{\partial y'} \left( \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial y'^2} \right) \]

\[ = \frac{\partial^2 \psi'}{\partial x'^4} + 2 \frac{\partial^2 \psi'}{\partial x'^2 \partial y'^2} + \frac{\partial^2 \psi'}{\partial y'^4} + Ha \frac{\partial^2 \psi'}{\partial x'^2} - Gr \frac{\partial T}{\partial x'} \quad \text{(7.a)} \]

Also, substituting (5.a,b) into (4.c) we obtain

\[ \frac{\partial \psi'}{\partial x'} \frac{\partial T'}{\partial y'} - \frac{\partial \psi'}{\partial y'} \frac{\partial T'}{\partial x'} = \mu_0 L \left( \frac{\partial^2 T'}{\partial x'^2} + \frac{\partial^2 T'}{\partial y'^2} \right) \quad \text{(7.b)} \]

where \( Ha, Gr \) and \( Pr \) are the Hartmann, Grashof and Prandtl numbers, respectively. They are defined as

\[ Ha = B_0 L \sqrt{\frac{\sigma}{\mu_0}} \quad Gr = \frac{\mu_0 L H_0 (T_b - T_c)}{\nu^2} \quad Pr = \frac{\nu}{\alpha_t} \quad \text{(8.a-c)} \]

where \( B_0 \) is the steady externally applied magnetic field of reference.
3. Test problem formulation

The test problem analyzed here is a square cavity, where the left wall is subjected to a hot temperature and the right wall is subjected to a cold temperature. The top and bottom walls are kept thermally insulated. All four boundaries are subjected to no-slip boundary conditions and a constant magnetic field is applied in the x direction (from the left to the right wall). Notice that the bi-harmonic equation (7.a) needs two boundary conditions for each wall, which are given along the ones for Eq. (7.b)

\[
\psi = \frac{\partial^2 \psi}{\partial x^2} = 0; \quad T = 1 \quad \text{at} \quad x = 0
\]  
\[
\psi = \frac{\partial^2 \psi}{\partial y^2} = T = 0 \quad \text{at} \quad x = 1
\]  
\[
\psi = \frac{\partial \psi}{\partial y} = \frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0
\]  
\[
\psi = \frac{\partial \psi}{\partial x} = \frac{\partial T}{\partial x} = 0 \quad \text{at} \quad y = 1
\]  

Several Hartmann numbers will be analyzed for two values of the Grashof number (Gr = 10^4 and Gr = 10^6) and the final results will be compared against Ref. [10] where the authors used the control volume method on a uniform grid of 40 × 40 grid cells. The Prandtl number was equal to 0.71 in all test cases.

4. Radial basis function approximation

Classical numerical methods, such as the finite volume method and the finite difference method, need to use some kind of pressure–velocity coupling scheme (for example, SIMPLEC Scheme [11]) in order to obtain velocity fields in the momentum equation (4.b) that satisfies the mass conservation equation (4.a). On the other hand, in such methods, the use of the bi-harmonic equation (7.a), which eliminates the pressure gradient, brings several problems of truncation error in the derivative approximations of the 4th order derivatives. Also, the convective terms are usually treated by some sort of hybrid or upwind method, such as the WUDDS [12] and the UTOPIA [13] algorithms.

In this paper, we use a RBF formulation to solve the Eqs. (7.a-b) as well as the boundary conditions (9.a-l). In the RBF formulation, the stream function and the temperature are written as

\[
\psi(x, y) = \sum_{i=1}^{M} \eta_i \xi_i(r_i)
\]  
\[
T(x, y) = \sum_{j=1}^{N} \lambda_j \xi_j(r_j)
\]

where the RBFs \( \xi \) are the same for the two expansions, but the parameters \( \eta \) and \( \lambda \) are different for each one. In Eqs. (10.a,b) and M and N are the number of centers used in the two RBF approximations.

Substituting Eqs. (10.a,b) into Eqs. (7.a,b) we can obtain

\[
\begin{align*}
\sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial x} \right] + \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial y} \right] + \sum_{i=1}^{M} \left[ \eta_i \frac{\partial^2 \xi_i(r_i)}{\partial x^2} \right] \\
- \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial x} \right] + \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial y} \right] + \sum_{i=1}^{M} \left[ \eta_i \frac{\partial^2 \xi_i(r_i)}{\partial x^2} \right] \\
= \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial x} \right] + 2 \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial y} \right] + \sum_{i=1}^{M} \left[ \eta_i \frac{\partial^2 \xi_i(r_i)}{\partial x^2} \right] \\
+ Ha^2 \sum_{i=1}^{M} \left[ \eta_i \frac{\partial^2 \xi_i(r_i)}{\partial x^2} \right] - Gr \sum_{i=1}^{M} \left[ \eta_i \frac{\partial \xi_i(r_i)}{\partial y} \right]
\end{align*}
\]

Note that the boundary conditions given by Eqs. (9.a-l) should also be written in terms of the RBFs. Thus, substituting (10.a,b) into Eqs. (9.a-l) we obtain

\[
\sum_{i=1}^{M} \eta_i \xi_i(r_i) = \sum_{i=1}^{N} \eta_i \xi_i(r_i) = 0; \quad \sum_{j=1}^{N} \lambda_j \xi_j(r_j) = 1 \quad \text{at} \quad x = 0
\]

Table 1

<table>
<thead>
<tr>
<th>Gr = 10^4</th>
<th>CPU time (s)</th>
<th>CPU time (s)</th>
<th>CPU time (s)</th>
<th>CPU time (s)</th>
<th>CPU time (s)</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RBF (6 × 6)</td>
<td>RBF (8 × 8)</td>
<td>RBF (15 × 15)</td>
<td>FVM (15 × 15)</td>
<td>FVM (15 × 15)</td>
<td>FVM (41 × 41)</td>
</tr>
<tr>
<td>Ha = 0</td>
<td>0.72</td>
<td>2.05</td>
<td>50.60</td>
<td>47</td>
<td>440</td>
<td></td>
</tr>
<tr>
<td>Ha = 10</td>
<td>0.66</td>
<td>1.70</td>
<td>34.03</td>
<td>56</td>
<td>560</td>
<td>652</td>
</tr>
<tr>
<td>Ha = 15</td>
<td>0.63</td>
<td>1.83</td>
<td>40.14</td>
<td>63</td>
<td>684</td>
<td></td>
</tr>
<tr>
<td>Ha = 25</td>
<td>0.56</td>
<td>1.69</td>
<td>42.59</td>
<td>69</td>
<td>684</td>
<td></td>
</tr>
<tr>
<td>Ha = 50</td>
<td>0.42</td>
<td>1.17</td>
<td>25.53</td>
<td>74</td>
<td>778</td>
<td></td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Gr = 10^6</th>
<th>Average Nusselt number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref. [10]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Ha = 0</td>
<td>2.01</td>
</tr>
<tr>
<td>Ha = 10</td>
<td>1.69</td>
</tr>
<tr>
<td>Ha = 25</td>
<td>1.14</td>
</tr>
<tr>
<td>Ha = 50</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Eqs. (11)–(13) result in a non-linear system of algebraic equations which can be solved with well-established numerical procedures such as the Broyden’s quasi-Newton method \[14\]. Several different choices are possible for the RBF function \[6\]. We used the multiquadric which is given as

\[
\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}
\]

where \(c\) is a shape parameter used to control the smoothness of the RBF. Up to this time, there is no well-established methodology to choose this shape parameter, although some empiricism can be found in the literature \[15,16\]. In this work, we will use the procedure defined by \[17\], where \(c\) is taken as the minimum distance between two center points over the entire domain. Thus, \(c\) is increased monotonically until the residual of the solution of Eqs. (7.a,b, 9.a-l) is minimum. This procedure implies solving very costly non-linear equations several times (in this paper we limited the upper value of \(c\) to ten times its initial value), but the final result is worthy.

Note that Eqs. (11) and (12) should be written for each collocation point inside the domain. Thus, if we have \(L\) collocation points, Eqs. (11) and (12) should lead to \(L\) equations each. Also, Eqs. (13.a-l) should be written for each collocation point at the boundaries. If we

Fig. 1. Velocity and temperature profiles for \(6 \times 6\) RBF centers.

Fig. 2. Velocity and temperature profiles for \(15 \times 15\) RBF centers.
In this paper we considered only the case where the cavity is not tilted. Two test cases correspond to the Grashof numbers equal to $Gr = 10^4$ and $Gr = 10^6$, where several Hartman numbers were analyzed. In all test cases, $Pr = 0.71$. All test cases presented in this paper were run on an Intel Centrino Duo (T2300 @ 1.66 GHz) processor with 1 Gb of RAM memory. The code was written in FORTRAN90 and the "cpu_time" intrinsic function was used to measure the computing time.

For $Gr = 10^4$, the following Hartman numbers were analyzed: $Ha = 0, 10, 25$ and $50$. Table 1 shows the computing time required to solve such problems, using several different numbers of RBF centers (collocation points) for the RBF approximation and different number of volumes for the finite volume method. Although the RBF approach does not require the use of a computing grid (the points can be randomly distributed), for the first results we used a uniformly distributed grid. It can be seen that the computing time is extremely low when solving such problems with the RBF. Although the RBF approximation with $15 \times 15$ collocation points needs approximately the same amount of computing time to solve the problem as the finite volume method with $15 \times 15$ volumes, Table 2 shows that the numerical error obtained with the RBF is an order of magnitude smaller. Also, in general, as the Hartman number increases, one can see from Table 1 that the computing time decreases for the RBF and increases for the finite volume method.

Table 2 shows the comparison for the average Nusselt number at the left and right walls computed in the present paper and in Ref. [10], for $Gr = 10^4$. This table also shows the results obtained by the finite volume method with primitive variables used in this work. From Table 2, it is clear that as we increase the number of centers in the RBF approximation, the discrepancy between these two sets of solutions becomes smaller. Ref. [10] only shows the average Nusselt number in graphics, so the ones used here for comparison were taken by measuring the graphics presented in [10] and are not necessarily error free. Keeping this in mind, the RBF-MHD formulation gives exceptionally good results, while requiring very small number of centers for its formulation. From the inspection of Table 1, most of the results were obtained in less than 30 s. A further investigation related to the choice of the shape parameter $c$ in the RBF approximation could reduce such computing time even more. From Table 2, one can verify that it is necessary to

<table>
<thead>
<tr>
<th>$Gr = 10^4$</th>
<th>CPU time (s)</th>
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<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RBF (15 x 15)</td>
<td>Ref [10]</td>
<td>RBF (25 x 25)</td>
<td>FVM (25 x 25)</td>
<td>FVM (41 x 41)</td>
</tr>
<tr>
<td>$Ha = 0$</td>
<td>134.80</td>
<td>1637.80</td>
<td>836</td>
<td>2593</td>
</tr>
<tr>
<td>$Ha = 10$</td>
<td>145.75</td>
<td>1288.08</td>
<td>919</td>
<td>2928</td>
</tr>
<tr>
<td>$Ha = 15$</td>
<td>138.00</td>
<td>2426.27</td>
<td>1014</td>
<td>3217</td>
</tr>
<tr>
<td>$Ha = 25$</td>
<td>166.58</td>
<td>1047.38</td>
<td>1058</td>
<td>3285</td>
</tr>
<tr>
<td>$Ha = 50$</td>
<td>73.09</td>
<td>913.36</td>
<td>1167</td>
<td>3223</td>
</tr>
<tr>
<td>$Ha = 100$</td>
<td>50.84</td>
<td>1528.84</td>
<td>1489</td>
<td>4560</td>
</tr>
<tr>
<td>$Ha = \infty$</td>
<td>71.52</td>
<td>4243.59</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

In order to show how accurate the RBF-MHD formulation can be, several test cases were analyzed, and the results were compared with Ref. [10] where the authors used the control volume method [18] on a grid of $41 \times 41$ equally spaced points. They solved the vorticity conservation equation and not the bi-harmonic approximation. Eq. (7.a). Also, in their paper, the results were shown for several different inclinations of the cavity. Since the original papers [10,18] did not show the computing times required to obtain the desired solution, we also solved these problems by the finite volume method with primitive variables [8] and compared the computing time required to obtain approximately the same order of accuracy for the solutions obtained using the RBF-MHD approximation.

5. Numerical results

In this paper we considered only the case where the cavity is not tilted. Two test cases correspond to the Grashof numbers equal to $Gr = 10^4$ and $Gr = 10^6$, where several Hartman numbers were analyzed. In all test cases, $Pr = 0.71$. All test cases presented in this paper were run on an Intel Centrino Duo (T2300 @ 1.66 GHz) processor with 1 Gb of RAM memory. The code was written in FORTRAN90 and the “cpu_time” intrinsic function was used to measure the computing time.

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Table 2 shows the comparison for the average Nusselt number at the left and right walls computed in the present paper and in Ref. [10], for $Gr = 10^4$. This table also shows the results obtained by the finite volume method with primitive variables used in this work. From Table 2, it is clear that as we increase the number of centers in the RBF approximation, the discrepancy between these two sets of solutions becomes smaller. Ref. [10] only shows the average Nusselt number in graphics, so the ones used here for comparison were taken by measuring the graphics presented in [10] and are not necessarily error free. Keeping this in mind, the RBF-MHD formulation gives exceptionally good results, while requiring very small number of centers for its formulation. From the inspection of Table 1, most of the results were obtained in less than 30 s. A further investigation related to the choice of the shape parameter $c$ in the RBF approximation could reduce such computing time even more. From Table 2, one can verify that it is necessary to
use a grid of 41 × 41 volumes in the finite volume method to obtain a result with approximately the same order of accuracy as the one with 15 × 15 centers in the RBF approximation. Thus, looking back at Table 1, one can see that the finite volume method requires at least one order of magnitude greater computing time than the RBF to obtain the same level of accuracy.

Fig. 1 shows the velocity profiles at the points located along the vertical axis and temperature along the horizontal axis of symmetry of the cavity, both for the RBF-MHD formulation with 6 × 6 centers and for the results presented in [10], for Gr = 10^4. One can see that even for this extremely low number of collocation centers, the velocity and the temperature profiles are very well captured. This becomes even more impressive when one looks at Table 1 and sees that such results were obtained in less than 1 s.

Fig. 2 shows the same results, but now using 15 × 15 RBF centers. One can see that the accuracy of the results improves and now the velocity profiles match the ones presented in [10], except for our results with Ha = 25. Actually, our results for Ha = 50 match results for Ha = 25 in [10]. This suggests that results labeled Ha = 25 in [10] should actually have been labeled Ha = 50.

Finally, Fig. 3 shows the isotherms and stream functions for Gr = 10^4, where it is clear that the MHD-RBF formulation gives very good results compared with the ones presented in [10], even for a very small number of centers.

The second set of test cases involved a Grashof number Gr = 10^6, which corresponds to conditions where the thermal buoyancy effects are two orders of magnitude stronger. Thus, the magnetic field needed to suppress such natural convection must be stronger than the one previously discussed for Gr = 10^4. For this higher Grashof number, the following Hartmann numbers were analyzed: Ha = 0, 10, 15, 25, 50, 100 and infinity. Table 3 shows the computing time required, using different number of RBF centers and also using different number of finite volumes in the finite volume method used in this work. For this set of test cases, with higher Grashof numbers, the number of required centers was greater than in the previous case with Gr = 10^4. Also, comparing Tables 1 and 3, one can see that, even for the same number of collocation points (15 × 15), the computing time for the RBF, in the case with higher Grashof number, increases by a factor greater than two for most Hartmann numbers. In general, when the Hartmann number increases, the computing time for RBF decreases. On the other hand, when the Hartmann number increases, the computing time for the finite volume method increases.

Table 4 shows the comparison for the average Nusselt number for the test cases where Gr = 10^6, taking approximately 30 min of computing time when using the RBF-MHD model to solve the non-linear bi-harmonic and energy equations. Comparing the RBF-MHD method with the finite volume method, one can see that for the same order of accuracy, the RBF-MHD required 15 × 15 collocation centers, while the finite

Table 4

<table>
<thead>
<tr>
<th>Gr = 10^6</th>
<th>Average Nusselt number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref. RBF Error FVM Error</td>
</tr>
<tr>
<td></td>
<td>(15 × 15) (%) (25 × 25) (%) (41 × 41) (%)</td>
</tr>
<tr>
<td>Ha = 0</td>
<td>8.76 10.68 21.92 9.21 5.14 8.24 5.94 7.98 8.90</td>
</tr>
<tr>
<td>Ha = 10</td>
<td>8.66 10.44 20.55 9.04 4.39 8.17 5.66 7.88 9.01</td>
</tr>
<tr>
<td>Ha = 25</td>
<td>8.04 8.92 10.95 8.32 3.48 7.81 2.86 7.39 8.08</td>
</tr>
<tr>
<td>Ha = 100</td>
<td>3.76 3.95 5.05 3.54 5.85 4.97 32.18 4.27 13.56</td>
</tr>
<tr>
<td>Ha = ∞</td>
<td>0.96 0.88 8.33 0.90 6.25 – – – –</td>
</tr>
</tbody>
</table>

---

**Fig. 4.** Velocity and temperature profiles for 25 × 25 RBF centers.
volume method needed 41 \times 41 finite volumes. Thus, from Table 3, the finite volume method required again a computing time that is at least one order of magnitude greater than the RBF-MHD model to obtain the same level of accuracy.

Fig. 4 shows the velocity and temperature profiles at mid-location of the cavity, both for the RBF-MHD formulation with 25 \times 25 centers, and for the results presented in [10], for Gr = 10^4. Although the isothersms are close to the results presented in [10], the peak values of the velocities are not very well captured, showing that, for this test case 25 \times 25 collocation points are not enough to solve such a strong natural convection problem. This indicates that more collocation points were needed for this test case. However, since the computing time would increase considerably, this was not conducted in this paper.

The streamlines and isothersms obtained by the RBF-MHD formulation behave in the same way as those obtained in [10], although some discrepancies were observed for the lower values of Hartmann numbers, where the mass, momentum and energy equations are more coupled.

In order to improve the accuracy of the solution, some extra results were obtained using a non-uniform distribution of centers. Four different distributions were used, where a parameter \( \chi \) was used to control such non-uniformity. Fig. 5 shows the distribution of the 10 \times 10 centers, for different values of the parameter \( \chi \).

Also, the possibility of using a pseudo-random distribution of collocation points was investigated. For such distribution, we used the Sobol’s [19] algorithm with the constraint that the minimum distance between any two points should meet some uniformity criteria. Thus, the RBF-MHD model could be easily extended to very complex geometries, where the grid generation is an issue. Fig. 5 also shows the distribution of 10 \times 10 centers, using such pseudo-random algorithm.

Table 5 shows the deviation of the average Nusselt number from the ones reported in Ref [10] for Gr = 10^4 and using 6 \times 6 centers, for different values of parameter \( \chi \) and also for a pseudo-random distribution of collocation points. From this table, the use of a non-uniform distribution of centers, with \( \chi = 0.4 \), reduces significantly the deviation from the results presented in [10]. As for example, the relative error for Ha = 0 decays from 22.39% to 5.97%. For Ha = 10 and 25 there is also a decay in the relative error, although for Ha = 50, the error increases a little. The use of a pseudo-random distribution for the collocation points also reduces significantly the error, except for the case where Ha = 0. In fact, for Ha = 50, the results obtained through this random distribution presented the lowest errors when compared to the values presented in [10]. The results when using the pseudo-random distribution of points are actually arithmetic means of results obtained on five different pseudo-randomly distributed sets of points obtained with different input parameters to Sobol’s algorithm.

When more centers are used (15 \times 15, instead of 6 \times 6), the deviation of the average Nusselt number from the ones reported in Ref [10], for different values of parameter \( \chi \) and for the pseudo-random distribution of the collocation points are presented in Table 6. For this test case, where the number of centers was already large, there was not much decrease of the relative error compared to the case where a uniform distribution of centers was used. From this Table, the best non-uniformity factor found was \( \chi = 0.2 \), where only the cases with Ha = 25 and 50 had a decrease in the relative error. Also, the pseudo-random distribution of the collocation points reduced the error to the same levels obtained with \( \chi = 0.2 \).

![Fig. 5. Distribution of the 10 \times 10 centers, for different values of the parameter \( \chi \) and for a pseudo-random distribution of the collocation points.](image)

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>RBF Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>5.97%</td>
</tr>
<tr>
<td>0.4</td>
<td>1.76%</td>
</tr>
<tr>
<td>0.6</td>
<td>0.87%</td>
</tr>
<tr>
<td>0.8</td>
<td>4.00%</td>
</tr>
</tbody>
</table>

Table 5 Influence of the non-uniformity factor \( \chi \) on the relative error of the solution for 6 \times 6 centers and the solution using a pseudo-random distribution of the centers.

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>RBF Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.76%</td>
</tr>
<tr>
<td>0.4</td>
<td>0.87%</td>
</tr>
<tr>
<td>0.6</td>
<td>4.00%</td>
</tr>
<tr>
<td>0.8</td>
<td>4.00%</td>
</tr>
</tbody>
</table>

Table 6 Influence of the non-uniformity factor \( \chi \) on the relative error of the solution for 15 \times 15 centers and the solution using a pseudo-random distribution of the centers.
Thus, although the non-uniformity factor presented some oscillatory behavior, the pseudo-random distribution was stable and capable of producing good results. For complex geometries, where the grid generation can be difficult, such approach can be very useful.

6. Conclusions

This work used the radial basis function formulation to solve a magnetohydrodynamic problem in two dimensions in an incompressible, steady-state and laminar flow-field with constant magnetic field applied. The RBF results were compared against control volume method results reported in the literature. We also compared the computing time with a finite volume method using primitive variables. The accuracy of RBF was very good and computing time was at least an order of magnitude smaller. The use of the RBF-MHD formulation in solving complex physical problems seems to be very promising, since it does not require a structured mesh generation. Even for partial differential equations of high order, such as the one used in this paper, they do not suffer from the classical truncation error presented in the finite difference or finite volume methods. The use of a pseudo-random distribution of the collocation points showed good results. Thus, the RBF-MHD model can be used in complex geometries, where the grid generation is difficult. The RBF formulation, however, needs more research in order to specify the best shape parameter in less computing time.

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References